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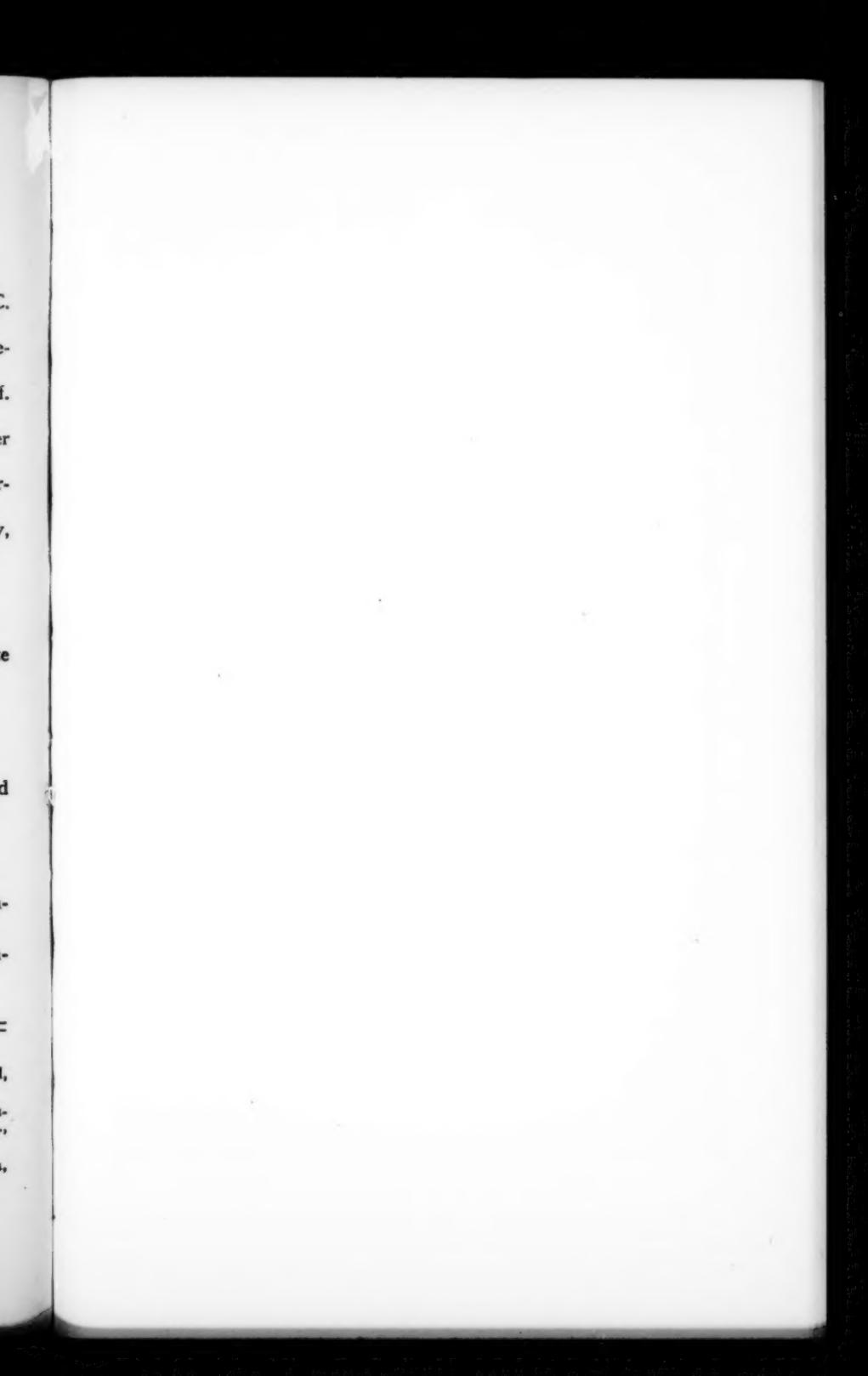
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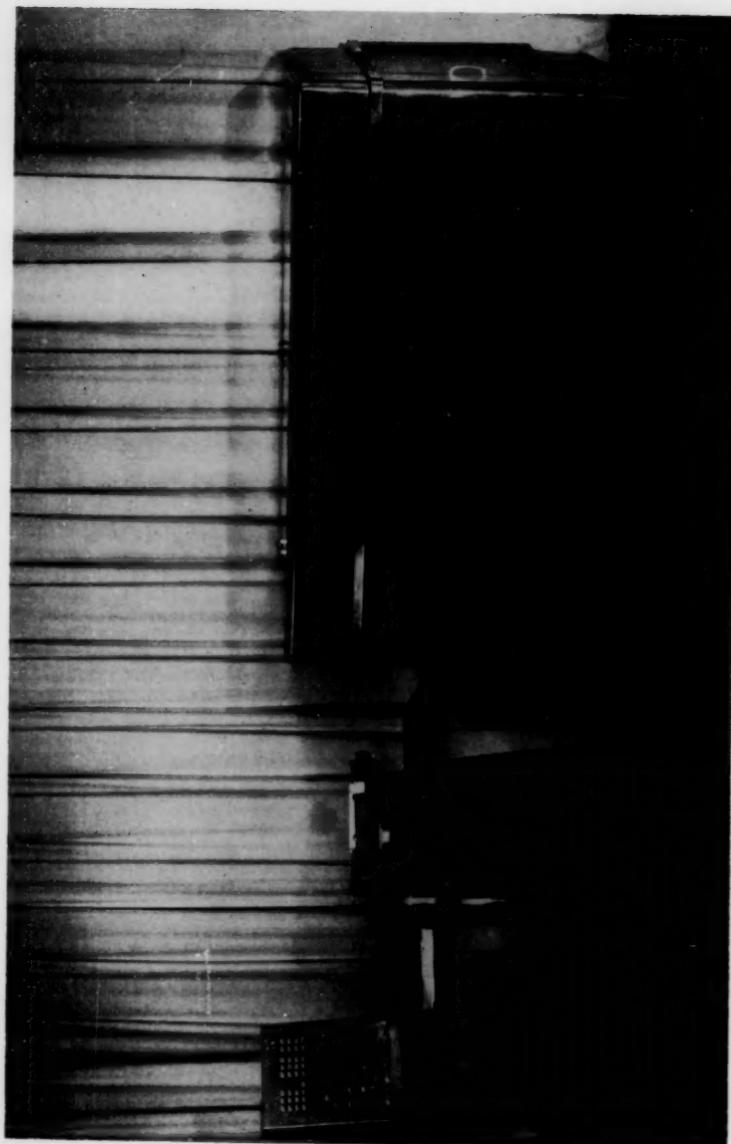
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THE CIRCLE COMPUTER



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Computing Eigenvalues and Eigenvectors of a Symmetric Matrix on the ILLIAC

One of the programs in the library of programs for the University of Illinois' electronic digital computer, known as the ILLIAC, is a program for finding the eigenvalues and eigenvectors of a symmetric matrix. The iterative method used is the rotation of axes method discussed by H. H. GOLDSFINE¹ in an unpublished paper and referred to by TAUSKY & TODD² as JACOBI's method.³ It consists essentially of performing a sequence of orthogonal transformations on the matrix, where each transformation is designed to reduce a selected off-diagonal element to zero. GOLDSFINE¹ shows that the sum of the squares of the off-diagonal elements is reduced, during a single transformation, by the amount $2a_{jk}^2$ (a_{jk} is the element reduced to zero by the transformation) and that the process produces a sequence of matrices whose limit is a diagonal matrix. His expression for an upper bound on the number of transformations required to diagonalize an n th order matrix is $[\ln(t_0/t_i)](n^2 - n)/2$, where t_0 is the sum of the squares of the off-diagonal elements of the original matrix and t_i is the same quantity after the i th transformation. Here it is assumed that a_{jk} is always greater than the average off-diagonal element in absolute value. Our results so far indicate that this bound is from ten to twenty times greater than the number actually required by our program. The eigenvectors are obtained by multiplying together the orthogonal matrices used in the successive transformations.

When the eigenvalues are not close, and the element a_{jk} is small, reduction of a_{jk} to 0 leaves the other elements unchanged in the first approximation. For the angle of rotation is given by the relation

$$\tan 2\phi = 2a_{jk}/(a_{jj} - a_{kk}).$$

When the off-diagonal elements are very small ϕ is of the order of a_{jk} ($a_{jj} \neq a_{kk}$). Now off-diagonal elements are transformed by

$$\begin{aligned} a_{rj} &= a_{rj} \cos \phi + a_{rk} \sin \phi = a_{rj}(1 - \phi^2/2! + \dots) + a_{rk}(\phi - \phi^3/3! + \dots) \\ a_{rk} &= -a_{rj} \sin \phi + a_{rk} \cos \phi = a_{rj}(-\phi + \phi^3/3! - \dots) \\ &\quad + a_{rk}(1 - \phi^2/2! + \dots) \end{aligned}$$

and hence are unchanged if second order terms are neglected. Thus one sweep through the off-diagonal elements reduces them to zero (up to terms of second order).

The purpose of this paper is to display some results of an investigation into the relative merits of

- (1) two approaches to the problem of how to select the off-diagonal element a_{jk} mentioned above, and
- (2) two approaches to the problem of when to apply the convergence test so as to terminate the process after convergence.

The two approaches mentioned in (1) are

- (a) to select the off-diagonal elements in sequence along successive rows of the matrix, and
- (b) to select the largest off-diagonal element each time.

The two approaches mentioned in (2) are

- (c) to apply the convergence test after each transformation, and

- (d) to apply the convergence test after each group of $(n^2 - n)/2$ transformations, where n is the order of the matrix. The convergence test used was a test of the size of t_i using double precision.

Method (a) is the simplest to program for an electronic computer but will require more transformations for convergence than method (b). Thus the accumulated round-off errors should be less using method (b). Method (c) enables one to terminate the process as soon as the process converges, but requires many applications of the test. Method (d) applies the test only after going through the off-diagonal elements once. This means that overiterating will result and in the extreme case $(n^2 - n - 2)/2$ unnecessary transformations will be performed.

The library program mentioned in the first paragraph is called program 42. It uses methods (a) and (d) and requires a total of 190 storage locations in the memory. Two modifications of this program have been written which are slightly longer. Program 42A uses methods (a) and (c) and program 42B uses methods (b) and (c). Tables 1-5 display the results obtained using these three programs on seven matrices of each of the orders 20, 16, 12, 8, and 4. They contain

- A. The time required to diagonalize each matrix,
- B. The number of orthogonal transformations required, and
- C. An indication of the accuracy.

Tables 6-9 contain the eigenvalues of the thirty-five matrices. In order to conserve space only five decimal places are included.

The time referred to in A is merely the computation time and does not include the time required for input or output of data. The accuracy of the process (item C above) is determined by forming the sum of the squares of the components of the n residual vectors,

$$r_i = Ax_i - \pi_i x_i \quad i = 1, 2, \dots, n$$

where x_i and π_i are the eigenvectors and eigenvalues, respectively, of A. This sum of squares is small and is scaled by 2^{30} before being printed.

Of the thirty-five matrices used in this investigation five were correlation matrices which were available (numbers 1, 8, 15, 22, and 29) and the remaining thirty were matrices generated by the machine. The method employed to generate the elements of these matrices was to square a number and use the middle digits of the product. Each new number then was used to generate the following number. The computation times for diagonalizing the five correlation matrices are slightly longer than those for the machine generated matrices due to the fact that certain changes were made in the ILLIAC, just after the five correlation matrices were diagonalized, which increased the speed of certain arithmetic operations.

Several conclusions can be drawn from an inspection of the results. The simplest program (number 42) using methods (a) and (d) was the fastest despite the fact that it over-iterated. However program 42B, using method (b), was in general the most accurate in the sense that the sum of squares of residuals was smallest. Obviously, the method (d) is superior to method (c). Program 42 never required more than seven sweeps through the off-

diagonal elements, i.e., no more than $7(n^2 - n)/2$ transformations were required for convergence. It appears that method (a) required about one and one-half times as many transformations as method (b).

Matrix	Time Min.	Time Sec.	Number of Transformations	$\frac{n}{2^n} \sum r_{ij}^2$
PROGRAM 42				
1	6	39	1330	.00 345
2	5	57	1330	.00 331
3	5	56	1330	.00 286
4	5	56	1330	.00 310
5	5	6	1140	.00 251
6	5	56	1330	.00 305
7	5	6	1140	.00 296
PROGRAM 42A				
1	15	18	1154	.00 272
2	14	25	1161	.00 279
3	14	27	1164	.00 282
4	14	17	1149	.00 326
5	13	32	1087	.00 230
6	14	23	1159	.00 314
7	13	56	1122	.00 263
PROGRAM 42B				
1	19	34	683	.00 145
2	19	19	685	.00 138
3	19	32	692	.00 131
4	19	32	692	.00 167
5	19	18	684	.00 148
6	19	17	684	.00 141
7	19	13	681	.00 165

TABLE 1

Matrix	Time Min.	Time Sec.	Number of Transformations	$\frac{n}{2^n} \sum r_{ij}^2$
PROGRAM 42				
8	3	28	840	.00 191
9	3	6	840	.00 163
10	3	6	840	.00 149
11	3	5	840	.00 142
12	2	39	720	.00 131
13	2	39	720	.00 104
14	2	39	720	.00 152
PROGRAM 42A				
8	6	54	726	.00 163
9	6	23	724	.00 126
10	6	23	725	.00 106
11	6	21	721	.00 133
12	5	53	667	.00 126
13	5	50	663	.00 106
14	5	37	637	.00 154
PROGRAM 42B				
8	8	10	425	.00 063
9	8	0	432	.00 066
10	7	53	426	.00 061
11	8	4	436	.00 070
12	7	47	421	.00 063
13	7	54	427	.00 072
14	7	57	430	.00 059

TABLE 2

COMPUTING EIGENVALUES AND EIGENVECTORS

Matrix	Time Min.	Time Sec.	Number of Transformations	$\frac{n=12}{2^{30} \sum r_{ij}^2}$
PROGRAM 42				
15	1	17	396	.000 600
16	1	9	396	.000 591
17	1	9	396	.000 447
18	1	9	396	.000 450
19	1	9	396	.000 502
20	1	9	396	.000 579
21	1	9	396	.000 600
PROGRAM 42A				
15	2	10	344	.000 681
16	1	59	340	.000 492
17	1	58	338	.000 506
18	1	58	339	.000 369
19	2	2	351	.000 407
20	2	2	351	.000 501
21	2	1	347	.000 631
PROGRAM 42B				
15	2	33	217	.000 233
16	2	41	239	.000 307
17	2	33	228	.000 277
18	2	35	230	.000 233
19	2	25	216	.000 256
20	2	32	227	.000 257
21	2	39	236	.000 232

TABLE 3

Matrix	Time Min.	Time Sec.	Number of Transformations	$\frac{n=8}{2^{30} \sum r_{ij}^2}$
PROGRAM 42				
22		20	140	.000 125
23		18	140	.000 134
24		18	140	.000 119
25		18	140	.000 118
26		18	140	.000 111
27		22	168	.000 165
28		18	140	.000 102
PROGRAM 42A				
22		28	126	.000 160
23		28	135	.000 113
24		26	129	.000 112
25		25	121	.000 106
26		28	134	.000 088
27		29	143	.000 208
28		28	135	.000 102
PROGRAM 42B				
22		35	97	.000 072
23		32	91	.000 047
24		30	88	.000 085
25		32	94	.000 063
26		32	93	.000 056
27		34	96	.000 043
28		34	97	.000 081

TABLE 4

Matrix	Time Min.	Sec.	Number of Transformations	$\frac{n=4}{2^{20} \sum_{ij}^2}$
PROGRAM 42				
29	2.5		24	.0000 212
30	2.4		24	.0000 095
31	3.0		30	.0000 070
32	2.4		24	.0000 075
33	2.3		24	.0000 137
34	2.4		24	.0000 110
35	2.4		24	.0000 086
PROGRAM 42A				
29	2.4		19	.0000 138
30	2.3		20	.0000 056
31	2.9		25	.0000 113
32	2.1		19	.0000 029
33	2.3		20	.0000 093
34	2.4		22	.0000 078
35	2.4		22	.0000 143
PROGRAM 42B				
29	2.4		17	.0000 055
30	2.3		16	.0000 051
31	2.4		16	.0000 053
32	2.4		16	.0000 030
33	2.4		17	.0000 042
34	2.4		16	.0000 067
35	2.4		19	.0000 090

TABLE 5

EIGENVALUES						
1	2	3	4	5	6	7
+.72951	+.70548	+.61214	-.60855	+.34710	+.65011	+.55872
-.75242	-.54004	-.38161	-.52812	+.56507	+.36704	-.71648
-.54106	-.50730	+.45455	-.43533	-.74842	-.73229	+.60475
+.41564	-.23097	-.72650	+.78575	+.54166	+.68674	-.33592
-.58256	-.64287	-.53514	+.46933	-.61324	+.57402	-.53408
-.28102	+.58653	-.43796	-.64543	-.27147	-.59379	+.69971
+.52031	+.26919	+.75447	+.27829	-.49548	-.23771	+.36562
+.58046	-.10555	+.55998	+.71874	-.52582	+.42814	-.37732
-.41469	+.29675	-.30252	-.32482	+.52150	-.49613	-.60475
-.29046	+.52572	+.12832	-.22613	-.45968	+.24109	+.29305
-.33468	-.37614	-.27717	+.59033	-.34006	+.34885	-.20574
+.54923	-.43715	-.07906	+.35744	-.10921	-.13801	+.27637
+.02789	+.48254	+.30587	+.07084	+.29205	-.46256	-.07809
+.18720	+.35102	+.48978	+.21038	-.08419	+.27996	-.00028
-.18534	-.16190	+.21471	-.11323	+.03378	+.05406	-.12379
+.13149	-.24689	-.23067	+.09255	+.18037	+.20607	+.41737
+.29696	+.06814	-.21203	+.02895	-.19826	-.41526	+.20575
+.05071	+.03994	+.23868	-.27643	+.00225	-.28314	-.28980
-.13306	+.17470	-.03577	-.06418	+.24630	-.05462	+.06703
-.05810	+.21225	+.02712	-.16923	+.07460	-.18670	+.13132

TABLE 6

EIGENVALUES

8	9	10	11	12	13	14
+.62833	+.64790	+.65658	-.54089	+.26411	+.42710	+.60556
-.55535	-.50376	-.45344	-.38215	+.49948	-.47585	+.13948
-.64081	-.43511	+.46503	-.51589	-.49670	+.57891	-.68004
+.39019	-.05532	-.61847	+.59733	+.48355	-.24169	+.55796
-.50721	-.57457	-.28238	+.47695	-.60610	-.59440	+.51185
-.33063	+.47444	-.36903	-.47087	-.39261	+.62438	-.54204
+.32001	+.17786	+.53729	+.07674	-.52051	+.15495	-.24080
+.51316	+.02452	+.35702	+.66319	-.33774	-.19197	+.31760
-.29364	+.20854	-.24871	-.14883	+.40512	-.32898	-.43488
-.13233	+.51314	+.16411	-.23124	-.15733	+.27840	-.03264
-.19180	-.32012	-.21132	+.36210	-.26548	-.31017	+.33752
+.24308	-.24645	-.01296	+.06539	-.12244	+.28416	-.17560
-.02683	+.33257	+.12396	+.19461	+.19613	+.00504	-.32918
+.17290	+.31621	+.26874	-.07243	+.01189	+.06621	+.26982
+.02344	-.12027	+.19060	-.08335	+.05150	-.07676	+.05161
+.10861	-.18881	-.10576	+.10039	+.13602	+.18048	-.09026

TABLE 7

EIGENVALUES

15	16	17	18	19	20	21
+.53175	+.46872	+.54439	-.44956	+.26122	+.38378	+.57151
-.54732	-.53186	-.38947	-.32509	+.44123	-.46303	+.05192
-.44538	-.33823	+.37032	-.30085	-.42957	+.51761	-.51638
+.33602	+.14008	-.50660	+.57480	+.24992	-.27931	+.45467
-.14756	-.40958	-.22885	+.29884	-.58716	-.48722	+.39733
-.16680	+.40960	-.33077	-.28068	-.32414	+.30840	-.30701
+.20161	+.17662	+.29323	+.08503	-.21857	+.09502	-.23531
+.35719	-.01891	+.18885	+.44081	-.30374	+.01101	+.12498
-.30020	+.27312	-.13267	-.11625	+.16273	-.11119	-.34457
+.04760	+.32131	+.21655	-.04640	-.02549	+.19796	-.10689
-.06842	-.20311	-.02120	+.21271	-.26387	-.20737	+.23361
+.12093	-.24408	+.05281	+.04666	+.03641	+.23940	-.09563

TABLE 8

EIGENVALUES

22	23	24	25	26	27	28
+.39745	+.39005	+.42526	-.30457	+.17009	+.21844	+.50218
-.37863	-.41997	-.34751	-.28376	+.34870	-.29635	+.08990
-.35548	-.14337	+.32709	-.12749	-.30897	+.28240	-.25758
+.22612	+.16712	-.27489	+.21208	+.13591	-.07422	+.32270
-.04768	-.25903	-.19429	+.29716	-.45210	-.32743	+.02900
+.02841	+.27762	-.03599	-.03213	-.26639	+.46710	-.39965
+.12765	+.06557	+.17299	+.07477	-.16227	+.03887	-.17113
+.06131	-.02717	+.09796	+.37703	-.09466	-.01919	-.05480
29	30	31	32	33	34	35
+.12263	+.18964	+.26306	-.28179	+.19975	+.18424	+.16572
-.32853	-.21833	-.26037	-.12093	-.01266	-.14856	-.07177
-.03726	-.02269	+.14103	+.08960	-.28262	+.28318	-.22650
+.19951	+.05038	-.14230	+.16096	-.21660	+.03009	+.12923

TABLE 9

¹ Institute for Advanced Study, Princeton, 1949.² O. TAUSKY & J. TODD, "Systems of equations, matrices and determinants," *Mathematics Magazine*, v. 26, 1952, p. 71-88.³ C. G. J. JACOBI, "Ein leichtes Verfahren, die in der Theorie der Säkularstörungen vorkommenden Gleichungen numerisch aufzulösen," *Jn. reine angew. Math.*, v. 30, 1846, p. 51-95.

A Numerical Analyst's Fifteen-Foot Shelf

Imagine a laboratory of numerical analysis, with computers, coders, problem analysts, and research mathematicians. The group varies in mathematical experience from new college graduates to professionals with long research records. Disappointingly few can profitably consult a book not in English but, pooling talents, the group can read English, French, German, Italian, Russian, and Hebrew quite well. Quite as important as a comprehensive library (assumed to lie within a few miles) is a small library in the laboratory building. What should such a working library contain?

Clearly the library needs a diversity of material, of which at least five classes can be distinguished:

- A. Mathematics books.
- B. Books on computing machines.
- C. Tables of functions.
- D. Periodicals.
- E. General references (e.g., language dictionaries).

This article is a proposed list of about 150 essential titles in class A. The reader is warned that the list has been hastily prepared and is very tentative; its inclusions and omissions should not be taken too seriously.

Class A was divided into the following five categories, of which four have been split into subcategories:

- 1. Bibliographies on mathematics.
- 2. Collections of formulas.
- 3. Books on numerical analysis.
- 4. Other books on applied mathematics.
- 5. Books on pure mathematics.

In selecting the titles five qualities were explicitly considered: (a) adequacy of material in topics likely to be needed; (b) use of English language; (c) completeness of bibliography; (d) readability; (e) recency. The order of precedence given to these qualities depended on the book user; for mature research men it was perhaps a, c, e, b, d, (most important first), while for junior computers, perhaps b, d, a, e, c. In categories 1, 2, and 3 a considerable proportion of the available books has been listed, so that any enlargement of the library would occur mainly in categories 4 and 5.

The bibliographical citations came from the books themselves, from the Library of Congress cards, or from Parke's very helpful *Guide* (see just below). Mrs. MILDRED MARTINOLICH helped prepare the citations.

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On Finding the Characteristic Equation of a Square Matrix

Various methods are known for finding explicitly the characteristic equation of a square matrix.^{1,2,3,4,5,6} Some of these make use of the Cayley-Hamilton theorem which states that every square matrix satisfies its own characteristic equation.^{3,4,5} In the present paper we describe among others,

the method of K. HESSENBERG,^{1,2} who uses the fact that similar matrices have the same characteristic equation. We also indicate a useful combination of Hessenberg's method with an iteration technique based upon the Cayley-Hamilton theorem. The Hessenberg method is as follows:

Let A be the given $n \times n$ matrix whose characteristic equation is sought. With an arbitrary column vector z_1 and certain scalars p_{ij} , to be defined presently, form the columns z_2, z_3, \dots, z_{n+1} in the manner

$$(1) \quad \begin{aligned} z_2 &= Az_1 + p_{12}z_1 \\ z_3 &= Az_2 + p_{13}z_1 + p_{23}z_2 \\ z_4 &= Az_3 + p_{14}z_1 + p_{24}z_2 + p_{34}z_3 \\ &\vdots \\ z_{n+1} &= Az_n + p_{1n}z_1 + p_{2n}z_2 + \cdots + p_{nn}z_n. \end{aligned}$$

Choose the p_{ij} so that the matrix Z has the form

$$(2) \quad Z = (z_1, z_2, \dots, z_n) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z_{22} & 0 & \cdots & 0 \\ 0 & z_{32} & z_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & z_{n2} & z_{n3} & \cdots & z_{nn} \end{bmatrix}$$

that is, the p_{ij} are taken such that z_1 has unity for its first component and zero for each of its other components, while the first k components of z_{k+1} vanish for $k = 1, 2, \dots, n - 1$.

Thus, defining the matrix P as

$$(3) \quad P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1,n-1} & p_{1n} \\ -1 & p_{22} & p_{23} & \cdots & p_{2,n-1} & p_{2n} \\ 0 & -1 & p_{33} & \cdots & p_{3,n-1} & p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & p_{nn} \end{bmatrix},$$

Hessenberg finds

$$(4) \quad AZ + ZP = 0.$$

For

$$(5) \quad AZ = (Az_1, Az_2, \dots, Az_n)$$

and if in (5) we replace Az_1, Az_2, \dots, Az_n by their equivalents as found from (1), we have

$$(6) \quad AZ = (z_2 - p_{12}z_1, z_3 - p_{13}z_1 - p_{23}z_2, z_4 - p_{14}z_1 - \cdots - p_{34}z_3, \dots).$$

On the other hand,

$$(7) \quad \begin{aligned} ZP &= z_1(p_{11}, p_{12}, \dots, p_{1n}) + z_2(-1, p_{22}, \dots, p_{2n}) + \cdots \\ &= (p_{11}z_1, p_{12}z_1, \dots, p_{1n}z_1) + (-z_2, p_{22}z_2, \dots, p_{2n}z_2) + \cdots \end{aligned}$$

and on adding (6) and (7) we obtain (4). Notice by (2) that the columns z_1, z_2, \dots, z_n are, in general, linearly independent. Thus when Z^{-1} exists,

from (4) it follows that

$$-P = Z^{-1}AZ$$

which shows that $-P$ is similar⁶ to A , and by a known theorem⁶ it follows that $-P$ has the same characteristic equation as A .

The characteristic equation is

$$\det(P + \lambda I) = 0, \text{ i.e.,}$$

$$\begin{vmatrix} (\lambda + p_{11}) & +p_{12} & +p_{13} & \cdots & +p_{1n} \\ -1 & (\lambda + p_{22}) & +p_{23} & \cdots & +p_{2n} \\ 0 & -1 & (\lambda + p_{33}) & \cdots & +p_{3n} \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ 0 & 0 & 0 & \cdots & (\lambda + p_{nn}) \end{vmatrix} = 0.$$

Expanding the determinant by the elements of its last column, one finds that the equation is

$$(8) \quad p_{1n} + p_{2n}F_1 + p_{3n}F_2 + \cdots + p_{n-1,n}F_{n-2} + p_{nn}F_{n-1} = 0,$$

where the F 's may be calculated by recursion:

$$(9) \quad \begin{aligned} F_1 &= \lambda + p_{11} \\ F_2 &= (\lambda + p_{22})F_1 + p_{12} \\ F_3 &= (\lambda + p_{33})F_2 + p_{23}F_1 + p_{13} \\ F_4 &= (\lambda + p_{44})F_3 + p_{34}F_2 + p_{24}F_1 + p_{14} \\ &\vdots \\ &\text{etc.} \end{aligned}$$

The matrices Z and P can be computed systematically.
Form the array:

A	Z
P	

It follows from (4) that the scalar product of any row of (A, Z) by any column of $\begin{bmatrix} Z \\ P \end{bmatrix}$ must vanish. If these scalar products are formed in proper succession, setting such a product equal to zero gives a simple equation containing just one "unknown", a z_{ij} or a p_{ij} , which is readily determined. Thus:

Multiplication of the first column of $\begin{pmatrix} Z \\ P \end{pmatrix}$

by the 1st row of (A, Z) permits determination of p_{11}
 by the 2nd row of (A, Z) permits determination of z_{22}
 by the 3rd row of (A, Z) permits determination of z_{32}
 by the 4th row of (A, Z) permits determination of z_{42}
 etc. etc.

Multiplication of the second column of $\begin{pmatrix} Z \\ P \end{pmatrix}$

by the 1st row of (A, Z) permits determination of p_{12}
 by the 2nd row of (A, Z) permits determination of p_{22}
 by the 3rd row of (A, Z) permits determination of z_{32}
 by the 4th row of (A, Z) permits determination of z_{42}
 etc.

Similarly, multiplication of each of the other columns of $\begin{pmatrix} Z \\ P \end{pmatrix}$ in proper succession by each of the rows of (A, Z) permits determination of all remaining elements.

After thus determining matrix P , the F 's may be computed from (9) and the characteristic equation by (8).

Exceptional Cases. As pointed out by ZURMÜHL,² minor difficulties sometimes occur. For instance, one of the z_{ik} may vanish when, according to the given procedure, it is used as a divisor to determine some element of P . Also an entire column of z_{ij} 's may consist of zeros. Treatment of such cases is indicated in the following examples.²

Case 1. Vanishing of a z_{ik} .

In Fig. 1 the work proceeds normally until we try to find p_{22} , where we have $2 + z_{22}p_{22} = 0$ which is meaningless since $z_{22} = 0$. However, this difficulty can be avoided in a way consistent with (4).

2	3	-2	1	0	0
0	1	2	0	0	2
1	2	-1	0	1	0
<hr/>					
			-2	2	-6
			-1	1	-4
			0	-1	-1

FIG. 1.

Assign z_{22} arbitrarily, say $z_{22} = 0$, and let s_{22} be "unknown." Then: Equating to zero the product of

Row 3 of (A, Z) by column 2 of $\begin{pmatrix} Z \\ P \end{pmatrix}$, determines $p_{12} = 1$,

Row 2 of (A, Z) by column 2 of $\begin{pmatrix} Z \\ P \end{pmatrix}$, determines $z_{22} = 2$,

Row 1 of (A, Z) by column 3 of $\begin{pmatrix} Z \\ P \end{pmatrix}$, determines $p_{13} = -6$,

Row 3 of (A, Z) by column 3 of $\begin{pmatrix} Z \\ P \end{pmatrix}$, determines $p_{23} = -4$,

Row 2 of (A, Z) by column 3 of $\begin{pmatrix} Z \\ P \end{pmatrix}$, determines $p_{33} = -1$.

Since the array of Fig. 1 now satisfies (4), and P has the standard form, we apply (9) and (8) to find the characteristic equation

$$\lambda^3 - 2\lambda^2 - 3\lambda + 2 = 0.$$

Alternatively, we could have assigned p_{22} arbitrarily, leaving z_{22} , z_{32} , p_{13} , p_{23} , p_{33} to be determined so that (4) holds. The resulting P will be similar to the one already found and the characteristic equation will be unchanged.

Case 2. Vanishing of a vector \mathbf{z}_k .

In Fig. 2 it happens that $\mathbf{z}_3 = \mathbf{0}$, apparently indicating that $p_{13} = p_{23} = p_{33} = 0$. Here we replace \mathbf{z}_3 by $\epsilon \mathbf{z}_3$ and let ϵ tend to zero in the final calculation.

$$\begin{array}{ccc|ccc} -3 & 1 & 3 & 1 & 0 & 0 \\ 10 & 0 & -6 & 0 & 10 & 0 \\ -10 & 2 & 8 & 0 & -10 & \epsilon \\ \hline & & & 3 & 20 & -3\epsilon \\ & & & -1 & -6 & 0.6\epsilon \\ & & & 0 & -1 & -2 \end{array}$$

FIG. 2.

Thus we find $p_{13} = -3\epsilon$, $p_{23} = 0.6\epsilon$, $p_{33} = -2$, and

$$F_1 = \lambda + 3$$

$$F_2 = (\lambda - 6)F_1 + 20 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

$$F_3 = (\lambda - 2)F_2 + 0 + 0 = (\lambda - 2)^2(\lambda - 1) = 0.$$

Evidently Case 2 occurs when the characteristic equation has a multiple root and A has linear elementary divisors corresponding to the root.

Iteration Techniques. While the use of (9) and (8) is a highly efficient way of getting the characteristic equation (8), it would appear that (9) and (8) are not as automatic as the method of finding P . To obtain a more fully automatic procedure, we propose that the Cayley-Hamilton theorem* be applied to the Hessenberg matrix P in the following way.

Let the characteristic equation of P be

$$(10) \quad c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n = 0$$

Then from the Cayley-Hamilton equation it follows that

$$(11) \quad c_0I + c_1P + c_2P^2 + \cdots + P^n = 0.$$

Post-multiply (11) by an arbitrary column matrix \mathbf{x}_0 . There results

$$(12) \quad c_0x_0 + c_1x_1 + c_2x_2 + \cdots + x_n = 0$$

where

$$(13) \quad P^k x_0 \equiv x_k.$$

If \mathbf{x}_0 is taken to be the column matrix $\{1, 0, 0, \dots, 0\}$, then (12) becomes in general a triangular system of linear equations from which the c 's are found in easy succession. The desired equation is finally obtained from (10) on replacing λ by $-\lambda$. If we define the matrices

$$X = (x_0, x_1, \dots, x_n),$$

$$C = \begin{bmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ c_{n-1} \end{bmatrix}$$

our complete scheme is represented by the array:

(14)	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td style="padding: 5px;">A</td><td style="padding: 5px;">Z</td><td></td></tr> <tr> <td style="padding: 5px;">P</td><td style="padding: 5px;">X</td><td style="padding: 5px;">C</td></tr> </table>	A	Z		P	X	C
A	Z						
P	X	C					

Thus, the procedure is fully automatic.

If the given matrix A has a sufficient number of zero elements in its lower left corner, and in particular if A is a continuant, then Hessenberg's Z and P are unnecessary in our method since we obtain a much more rapid solution by direct application of the iteration technique, as indicated by the array:

(15)	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td style="padding: 5px;">A</td><td style="padding: 5px;">X</td><td style="padding: 5px;">C</td></tr> </table>	A	X	C
A	X	C		

In the cases where (15) applies, (10) directly represents the characteristic equation of A , so that here we need not replace λ by $-\lambda$.

Elementary Transformations. The elementary transformations or operations* which, applied either singly or jointly to the square matrix A , leave invariant the characteristic equation of A are as follows:

1. If the i -th and j -th rows are interchanged, the i -th and j -th columns must be interchanged.
2. If the elements of the i -th row are multiplied by k , the elements of the i -th column must be multiplied by $1/k$.
3. If k times the j -th row is added to the i -th row, then the negative of k times the i -th column must be added to the j -th column.

Using such elementary operations, we can always transform A so that it has an isosceles right-triangular array of $(n - 1)(n - 2)/2$ (= triangular number of order $n - 2$) zeros in its lower left corner, where n is the order of A , so that by using the scheme (15) on the new matrix we readily obtain the characteristic equation of A .

The presence of zero elements in A generally facilitates the transformation. Generally the transformation is not too laborious in any specific case and we recommend that the characteristic equation be found by applying scheme (15) to the transformed matrix rather than by applying scheme (14) to the original matrix. A great deal will of course depend on the nature of the matrix and the kind of equipment available for computation.

Examples. Consider the matrix A_1 ,

$$A_1 = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

which was used in a previous illustration and with which there was some difficulty using the Hessenberg method.

By interchanging the second and third rows and then interchanging the second and third columns of A_1 , we obtain

$$T_1 = \begin{bmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

On applying scheme (15) to T_1 , we have

2	-2	3	1	2	2	8	2
1	-1	2	0	1	1	5	-3
0	2	1	0	0	2	4	-2

From the last column, we read that the characteristic equation of A_1 is

$$\lambda^3 - 2\lambda^2 - 3\lambda + 2 = 0.$$

An example of the complete procedure (14) for a matrix³ of fourth order is the following:

1	-2	3	-2	1	0	0	0
1	5	-1	-1	0	1	0	0
2	3	2	-2	0	2	1	0
2	-2	6	-3	0	2	2	-2
				-1	0	1	-4
				-1	-1	3	-2
				0	-1	2	0
				0	-1	-4	6
				0	0	-1	5

From the column C we infer

$$\det(P - I\lambda) = \lambda^4 + 5\lambda^3 + 6\lambda^2 - 4\lambda - 8 = 0.$$

Hence the characteristic equation of A is

$$\det(A - \lambda I) = \det(P + I\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = 0.$$

Number of Operations Required. An "operation," for the present purpose, is either a multiplication or a division of a pair of numbers.

1. (a) If we ignore multiplication by λ , such as $\lambda \cdot p_{22}$ etc., in the formation of the F 's, then to obtain the characteristic equation Hessenberg requires $M_1 = n^3 - (3/2)n^2 + (1/2)n$ multiplications and $D_1 = n(n-1)/2$ divisions.
- (b) If we include trivial multiplications such as $\lambda \cdot p_{22}$, then Hessenberg requires $M_2 = n^3 - (1/2)n^2 + (3/2)n$ multiplications and D_1 divisions.
2. If we use Hessenberg's method to obtain P and then apply the Cayley-Hamilton theorem to P , then to obtain the characteristic equation in this way we require $M_3 = n^3 - n^2$ multiplications and D_1 divisions.
3. If the matrix A has no zeros that may be taken advantage of, and elementary operations are used to obtain a similar matrix with $(n-1)(n-2)/2$ zeros in the lower left corner, and if the iteration scheme (15) is used starting with the column vector $(1, 0, 0, \dots, 0)$, then $(7/6)n^3 - 2n^2 + 17n/6 - 3$ multiplications and n divisions are required to find the characteristic equations. This indicates that, at least for large n , methods 1, 2, and 3 require nearly the same

- number of operations, while method 2 has the advantage of being most automatic. However, for special matrices or matrices having many zeros method 3 may be best.
4. The methods of FRAME⁸ or HOTELLING-BINGHAM-GIRSHICK⁹ require a number of operations of the order of n^4 since they depend upon $n - q$ multiplications of n by n matrices. Multiplication of two n by n matrices requires n^3 operations, and q is some constant independent of n , usually 1, 2, or 3.

Remarks. A use for the characteristic equation which does not seem to have been explicitly mentioned in the literature is the following. In many physical problems it is necessary to find the eigenvalues and eigenvectors of a square non-singular matrix. For this purpose iterative methods are generally applied. Quite often it is sufficient to find not all of the eigenvalues but only a few of the lowest ones. Since the method of iteration leads to the dominant eigenvalue, it has been necessary to find the inverse matrix to use the fact that the largest eigenvalue of the inverse matrix is the reciprocal of the smallest eigenvalue of the original matrix.

However, instead of finding the inverse of the original matrix, one may find the characteristic equation of the original matrix explicitly and at once write down the equation which has for its roots the reciprocals of the original roots. If (10) is the characteristic equation then

$$(16) \quad \mu^n + \frac{c_1}{c_0} \mu^{n-1} + \cdots + \frac{c_{n-1}}{c_0} \mu + \frac{1}{c_0} = 0$$

has the reciprocals of λ as its roots, which may be found by any convenient method.

The roots of (16) may also be found by matrix iteration methods applied to the *comparison matrix* or *companion matrix*:

$$\left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{1}{c_0} & -\frac{c_{n-1}}{c_0} & \cdots & -\frac{c_1}{c_0} & \end{array} \right]$$

which has (16) as its characteristic equation. Inversion of the original matrix is thus avoided, and incidentally the iteration procedure is easily carried out because of the large number of zeros in the matrix.

It is interesting to note that DANILEVSKI¹ has devised a method for finding the characteristic equation by reducing the matrix A to the companion matrix form by elementary transformations. However, this method does not compare favorably with the first three discussed above.

In conclusion we should like to point out that some of the difficulties encountered by Hessenberg are avoided by the method of obtaining zeros in the lower left hand corner and using iteration. One such example has already been shown. Consider now the other example, which had to be treated as a special case. If row 2 is added to row 3 and then column 3 is

subtracted from column 2, the matrix

$$A_2 = \begin{bmatrix} -3 & 1 & 3 \\ 10 & 0 & -6 \\ -10 & 2 & 8 \end{bmatrix}$$

becomes the similar matrix

$$T_2 = \begin{bmatrix} -3 & -2 & 3 \\ 10 & 6 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

Here obviously $(\lambda - 2)$ is a factor and all that remains to do is find the other factor from the matrix

$$\begin{bmatrix} -3 & -2 \\ 10 & 6 \end{bmatrix}$$

This is easily found to be $(\lambda - 2)(\lambda - 1)$.

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¹ FIAT *Review of German Science*, Applied Mathematics. Part I, 1948, p. 31-33.

² R. ZURMÜHL, *Matrizen*. Berlin, 1950, p. 316-322.

³ P. S. DWYER, *Linear Computations*. New York, 1951.

⁴ I. M. GEL'FAND, *Lektsii po Lineinoi Algebre* [Lectures on Linear Algebra]. Moscow 1951, p. 233-239.

⁵ H. E. FETTIS, "A method for obtaining the characteristic equation of a matrix and computing the associated modal columns," *Quart. Appl. Math.*, v. 8, 1950, p. 206-212.

⁶ G. BIRKHOFF & S. MACLANE, *A Survey of Modern Algebra*. New York, 1941.

RECENT MATHEMATICAL TABLES

1122[A,B,C].—P. P. ANDREEV, *Matematicheskie Tablitsy* [Mathematical Tables]. Moscow, 1952, 471 p. 12.5 × 19.7 cm. Price 7.75 rubles.

The main table of this work is a table of n^s for $n = 1(1)10000$, $s = -1/2, 3, 2, 1/2, 1/3$. Values are given to 6S only. The values of $(10n)^{1/2}$ are also given for $n > 1000$ while the natural logarithm of n is given for $n \leq 1000$. This part occupies 333 pages, only 30 values of n being devoted to each page. This table is certainly no substitute for BARLOW.

The second part of the volume is devoted to 19 small tables of minor importance including a 6S table of $1/n$ for $n = 1(1)10000$, $\log n$ for $n = 1(1)1000$ to 9D, and the binomial coefficients of the first 50 integral powers.

D. H. L.

1123[A].—M. LOTKIN & M. E. YOUNG, *Table of Binomial Coefficients*. Ballistic Research Laboratories Memo. Report No. 652. Aberdeen Proving Ground, 1953, 37 p., 21.6 × 27.9 cm. mimeographed from typescript.

The table gives

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

for non-negative integers r and for $n = 0(1)100$. Because of symmetry the table is for $r \leq (n+1)/2$. The values of the coefficients are to 20 significant figures. This means that exact values are available for $n \leq 69$ whereas only the first 19 values of $\binom{100}{r}$ are exact.

The table was prepared first in exact form and then abridged by rounding to 20 figures. An unabridged edition would have been of some value to number theorists who are concerned with congruence properties of binomial coefficients. The present table extends the table of PETERS & STEIN¹ from $n = 60$.

D. H. L.

¹ J. PETERS & J. STEIN, *Zehnstellige Logarithmentafel*, Band I. Berlin, 1922. Appendix, p. 69-74.

1124[A, B, C, D, E, F, L].—P. WIJDENES, *Noordhoff's Wiskundige Tafels in 5 Decimalen, Mathematical Tables to 5 Decimal Places*. Groningen, 1953, viii + 269 p., 23.8 × 17.1 cm. Price 8.75 florins.

This collection of tables contains unusually elaborate tables of the trigonometric functions such as a 5D table of cotes for angles up to 3 degrees at intervals of one second (taken from RHETICUS' *Canon Sinuum*).

Other tables not usually found in 5 decimal collections include a list of primes below 10000 and their natural logarithms, a factor table to 11197, a 6D table of e^x and e^{-x} for $x = .001(.001)1(.01)4$, and some minor tables of the factorial function, the sine, cosine, and exponential integrals, the error function, and the Bessel functions J_0 , J_1 , Y_0 , Y_1 . There are many conversion tables, some with real accuracy. The result is a neat, well-printed, and well-arranged handbook of useful tables.

The introduction and explanations are in 6 languages, including Spanish and Malayan.

D. H. L.

1125[A,B,C,D,F].—TSUNETA YANO, *Kokumin Suhyō [People's Tables]*. Tokyo, 1952, iv + 146 p., 12.1 × 17.8 cm. Price 230 yen.

This little handbook of useful tables includes the following:

1. p. 60-79—Multiplication table $A \cdot B = N$ for $A = 1(1)100$ and $B = 1(1)99$.
2. p. 80-81—Table of reciprocals, $100/N$, $N = 1(1)1000$ to 3D.
3. p. 82-83—Table of A^k , $k = 2, 3, 1/2, 1/3$, $A = 1(1)100$.
4. p. 84—Natural sines, cosines, and tangents for every degree of the first quadrant.
5. p. 85-93—Factor table giving the least prime factor of all numbers prime to 10 between 1000 and 10000.
6. p. 94-95—Table of the first nine multiples of primes ≤ 557 except 2, 5 arranged so that the final decimal digits are 1, 2, 3, ..., 9. With each p_n is given $p_{n+1}^2 - 1$. This table is intended to be used in testing a given number for small prime factors.

7. p. 96–115—Common logarithms, $\log N$ for $N = 1(1)999$ to 15D, $\log p$ for primes p between 1000 and 10000 to 7D, Δ ; $\log N$ for $N = 1(1)150$ to 7D, and $\log N$ for $N = 99900(1)99999$ to 15D, Δ , Δ^2 .
 8. p. 116–120—Antilogarithms to 4D, Cologarithms to 4D, and 6D tables of kM and k/M ($k = 1(1)100$), $M = \log e$.

The rest of the tables are non-mathematical, being tables of weights and measures, dates of emperors, latitudes and longitudes of Japanese cities and description of 12 wind intensities.

D. H. L.

1126[B,F].—D. R. KAPREKAR, *Cycles of Recurring Decimals (From $N = 3$ to 161 and some other numbers)*. Khare Wada, Deolali, India, 1950. Published by the author, vi + 55 + 2 p. 24.3 × 16.7 cm. Price 5 rupees.

This work contains tables of the complete periods in the decimal representation of M/N where the integers M and N are prime to each other and N is prime to 10 and does not exceed 163 (not 161 as implied in the title). Since the decimal representation of M/N is a cyclic permutation of that of many other rational numbers with the same denominator, it is possible to save much space by listing only those decimal representations which are not cyclic permutations of another. Thus for $N = 21$ there are the two cycles written as follows:

1	10	16	13	4	19	2	20	11	5	8	17
0	4	7	6	1	9	0	9	5	2	3	8

The first line gives the various numerators M and that cyclic permutation of the second line which begins with the digit under M gives the period of the decimal representation of M/N . Thus

$$13/21 = .619047619\cdots$$

In other words, the first line consists of the successive powers of 10 modulo $N = 21$.

The booklet contains reprints of several notes by the author on decimals. It should be useful for pencil computers who dislike long division.

D. H. L.

1127[C].—S. KHRENOV, *Semiznachnye Tablitsy Trigonometricheskikh Funktsii [Seven-place Tables of Trigonometric Functions]*. Moscow-Leningrad, 1951, 415 p., 21.6 × 29.2 cm. Price 33.75 rubles.

These tables give the natural values of the functions. All six functions are given to 7S for every 10 seconds of arc. This main table is preceded by a large 7S table of cotangents and cosecants for every second up to 10 degrees. Besides these two big tables there are 9 minor conversion tables for astronomical applications etc.

The page is large and well set out with ample room for first differences and the usual tables of proportional parts. The printing is somewhat irregular but perfectly legible and the paper is of better quality than one is used to in Russian publications.

Apparently this work is original although the author doesn't actually say so. No details of calculation are mentioned. He states that the tables were compared with those of GIFFORD, ANDOYER, and PETERS.

D. H. L.

1128[F].—K. GOLDBERG, "A table of Wilson quotients and the third Wilson prime" London Math. Soc., Jn., v. 28, 1953, p. 252-256.

Wilson's quotient is the integer

$$w_p = [1 + (p - 1)!] / p$$

where p is a prime. The author defines a Wilson prime as a prime p for which w_p is divisible by p . That 5 and 13 are Wilson primes has been known for a long time. MATHEWS¹ has asked for a rule to discover other Wilson primes. The author used the SEAC to show that below 10000 there is but one other, namely, $p = 563$. His table gives w_p reduced modulo p for all primes $p < 10000$.

Wilson's quotient occurs in the theory of symmetric functions modulo p , Bernoulli numbers and Fermat's last theorem. The previously published table of BEEGER² extended to $p = 300$. The unpublished table of WALL is for $p \leq 5381$ and is described in UMT 150, MTAC, v. 6, p. 238.

D. H. L.

¹ G. B. MATHEWS, *Theory of Numbers*. Cambridge 1892, New York, 1927, p. 318.

² N. G. W. H. BEEGER, "On the congruence $(p - 1)! \equiv -1 \pmod{p^3}$," *Messenger of Math.*, v. 49, p. 177-178, 1920.

1129[F].—SIGEKATU KURODA, "Über die Zerlegung rationaler Primzahlen in gewissen nicht-abelschen galoisschen Körpern," Math. Soc. Japan, Jn., v. 3, 1951, p. 148-156.

The decomposition of rational prime numbers is studied in certain non-abelian normal fields of degree 2^n . The laws for these decompositions have only been fully explored in the case of abelian fields and no complete extension of class field theory to non-abelian fields has yet been achieved. The fields K^* in question are of the form $K^* = R^*K$, where K is a field $R(i, \sqrt{\mu}, \sqrt{\bar{\mu}})$, where μ is an integer of $R(i)$ without square divisors, neither real nor purely imaginary and $\bar{\mu}$ is the complex conjugate of μ . The field R^* is composed of all quadratic extensions $R(\sqrt{l})$ where l runs through the prime factors of the discriminant of $R(\sqrt{\mu\bar{\mu}})$. Let $\mu = c\alpha$ where c is a positive rational integer and α has no rational divisor and put $a\bar{\alpha} = m$. The decomposition of the primes in K^* is linked up with the value of the biquadratic character symbol $(m/p)_4$.

The case $m = 65$ is discussed in detail and there is a table (p. 156) giving the value of the symbol $(65/p)_4$ for all 69 primes $p \leq 4549$ of the form $4x + 1$ which are quadratic residues of 5 and 13. There are two cases according as $p = x^2 + 65y^2$ or not. In the former case x and y are given.

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1130[F].—V. S. NANDA, "Tables of solid partitions," Nat. Inst. Sci. India, Proc., v. 19, 1953, p. 313-314.

The author tabulates the function $p^{(3)}(n, m)$ which enumerates the number of solid partitions of n in which the smallest part is m . Since

$$p^{(3)}(n, m) = 0 \quad \text{if } \frac{1}{2}n < m < n$$

and

$$p^{(3)}(n, n) = \frac{1}{2}n(n+1)$$

it suffices to tabulate the function for $m \leq \frac{1}{2}n$. This is done for $n = 1(1)25$.

For $m = 1$ we have

$$p^{(3)}(n, 1) = p^{(3)}(n - 1),$$

where $p^{(3)}(n)$ is the number of unrestricted solid partitions of n generated by

$$\begin{aligned} \prod_{r=1}^{\infty} (1 - x^r)^{r(r+1)/2} &= \sum_{n=0}^{\infty} p^{(3)}(n)x^n \\ &= 1 + x + 4x^2 + 10x^3 + \dots \end{aligned}$$

The reader is referred to a previous paper¹ for details and a small table of $p^{(3)}(n)$ for $n = 5(5)25$.

D. H. L.

¹ V. S. NANDA, "Partition theory and thermodynamics of multidimensional oscillator assemblies," Camb. Phil. Soc., Proc., v. 47, 1951, p. 591-601.

1131[F].—G. PALAMÀ & L. POLETTI, "Tavole dei numeri primi dell'intervallo 12012000-12072060," Unione Mat. Ital., Boll. s. 3, v. 8, 1953, p. 52-58.

This is a 6 page table giving the 3684 primes between the limits mentioned in the title. This list was prepared by hand using the "Neocribrum" of POLETTI.

1132[K].—E. E. SLUTSKIY, *Tablitsy dlia Vychisleniya Napolnoj Γ-funkcii i Funkcii Veroiatnosti x^2* [Tables for the Computation of the Incomplete Gamma Function and the probability function of x^2]. Edited by A. N. KOLMOGOROV. Leningrad, 1950, Acad. Nauk USSR, 71 p., 22.5 × 29.0 cm.

In the introduction the author states that the tables are intended for "finding the values of two integrals:

(1) The incomplete Gamma function

$$I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} x^p e^{-x} dx$$

and the probability function of x^2 :

$$(2) P(x^2, n) = \frac{1}{2^{(n-2)/2} \Gamma\left(\frac{n}{2}\right)} \int_x^{\infty} x^{n-1} e^{-\frac{1}{2}x^2} dx.$$

He lists three reasons why KARL PEARSON's, *Tables of the Incomplete Gamma Function*,¹ do not solve this problem completely. Then he continues:

"To remove the indicated shortcomings of K. Pearson's tables it has accordingly been necessary:

- 1) to construct tables for large values of n , up to $n = \infty$;
- 2) to construct satisfactory tables for small values of n , close to zero;
- 3) to facilitate interpolation to the highest possible degree.

It was of course impossible to meet all these requirements by any single construction. In view of this fact the tables are divided into parts."

The parts and their contents are as follows:

I $(\frac{1}{2}x^2)^{-n/2}(1 - P(x^2, n))$	$x^2 = 0(.05).2(.1)100$	$n = 0(.05).2(.1)6$
II (a) $P(x^2, n)$	$x^2 = 0(.1)3.2$	$n = 0(.05).2(.1)6$
(b) $P(x^2, n)$	$x^2 = 3.2(.2)7(.5)10(1)m$ $17 \leq m \leq 35$	$n = 0(.1).4(.2)6$
III $P(\frac{1}{2}(t + (2n)^{\frac{1}{2}})^2, n)$	$t = -4(.1)4.8$	$n = 6(.5)11(1)32$
IV $P(\frac{1}{2}(t + 2x^{-2})^2, 2x^{-2})$	$t = -4.5(.1)4.8$	$x = 0(.02).22(.01).25$

Table V gives the coefficients in the Everett and Newton interpolation formulas for an interval of .001.

All five tables are given to 5 decimal places except the Newton coefficients which are given to four places.

To facilitate interpolation the tables give the second and fourth central differences with respect to x^2 and t and the second central difference with respect to n and x . The introduction contains one section discussing methods of interpolating with respect to two variables in general, and another section giving practical instructions for the use of the tables. The latter section gives examples of the recommended interpolation methods.

As an evaluation of these tables, two statements made by Kolmogorov in the preface are probably correct. He says, "E. E. Slutskil's tables provide a firm basis for the preparation of every kind of simplified tables intended for lesser accuracy" and "As far as I know, such a comprehensive tabular representation of so complicated a function of two variables appears in mathematical literature for the first time. The labor of the last years of E. E. Slutskil's life, published after his death, has therefore an even wider methodological significance: it points the path to further work on the preparation of tables of great accuracy of functions of two variables."

In the reviewer's opinion, these tables do not provide the final answer to the tabulation of the incomplete Gamma function. In the first place, values are given to only five decimal places: this accuracy is insufficient for many present-day computing requirements. A second and more serious deficiency of the tables is that they do not provide for inverse interpolation. The reason for giving only five decimal places is that Slutskil computed most of his values by interpolation in Karl Pearson's seven place tables. The reason for not giving an inverse table probably is the computational complexity involved in computing it. It is to be hoped that makers of future tables of probability functions will consider the problem of inverse interpolation.

D. TEICHROEW

NBSINA

¹ K. PEARSON, *Tables of the Incomplete Γ-function*. London, 1934.

1133[L].—ADMIRALTY RESEARCH LABORATORY, "A solution of the equation $(y'')^2 = y y'$," A.R.L./T.1/Maths 2.7, 16 p.; "Table of $F(\beta, \rho)$," A.R.L./T.2/Maths 2.7, 10 p., Teddington, Middlesex, England.

These are the photostat tables referred to in RMT 1041, MTAC v. 6, 1952, p. 235–236, where F is denoted by f .

1134[L].—ADMIRALTY RESEARCH LABORATORY, "Tables of $F(x)$ and of $x^{-1} F(x)$," A.R.L./T.4/Maths. 2.7, 8 p., Teddington, England.

"The tables give 4D values of $F(x)$ and of $x^{-1} F(x)$,

$$F(x) = 2x \sum_{n=0}^{\infty} [x^2 + (2n+1)^2]^{-3/2}$$

for $x = 0(.002)1.5(.01)5$, and are not in error by more than .7 of a unit of the last place. Linear interpolation is adequate throughout."

1135[L].—ADMIRALTY RESEARCH LABORATORY, "Tables of $G(x)$," A.R.L./T.5/Maths. 2.7, 5 p., Teddington, Middlesex, England.

$$G(x) = 6x^2 \sum_{n=0}^{\infty} [x^2 + (2n+1)^2]^{-5/2}$$

is tabulated here to 4D for $x = 0(.002)1.5(.01)5$, with a maximum uncertainty of about two units of the last place. The table is linear."

1136[L].—ADMIRALTY RESEARCH LABORATORY, "Tables of $f(x)$ and of $xf(x)$," A.R.L./T.6/Maths. 2.7, 4 p., Teddington, Middlesex, England.

"The tables give 4D values of $f(x)$ and of $xf(x)$,

$$f(x) = \sum_{n=0}^{\infty} (-)^n (2n+1) [x^2 + (2n+1)^2]^{-5/2}$$

for $x = 0(.005)2(.05)5$, and are not in error by more than .7 of a unit of the last place. Both tables are linear."

1137[L].—W. E. BLEICK, *Tables of Associated Sine and Cosine Integral Functions and of Related Complex-Valued Functions*. Technical Report No. 10, U. S. N. Bureau of Ships, Monterey, 1953, 103 p. 20.3 × 26.7 cm. Mimeographed.

The author defines

$$(1) \quad Sia(x,y) = \int_0^x \frac{t \sin t}{t^2 + y^2} dt; \quad Cia(x,y) = \int_\infty^x \frac{t \cos t}{t^2 + y^2} dt$$

$$(2) \quad Si(x+iy) = \int_{0+iy}^{x+iy} t^{-1} \sin t dt; \quad Ci(x+iy) = \int_{\infty+iy}^{x+iy} t^{-1} \cos t dt.$$

In (2), the path of integration is to be taken parallel to the x -axis, and the branch cut is taken on the negative real axis, including the origin. Then it can be shown that

$$Sia(x,y) = \operatorname{Re} Si(x+iy) \cosh y + [\operatorname{Im} Ci(x+iy) - \pi/2] \sinh y$$

$$Cia(x,y) = \operatorname{Re} Ci(x+iy) \cosh y - [\operatorname{Im} Si(x+iy) + \frac{1}{2} \operatorname{Ei}(y) - \frac{1}{2} \operatorname{Ei}(-y)] \sinh y,$$

where

$$\text{Ei}(x) = \int_{-\infty}^x t^{-1} e^t dt$$

is the well known exponential integral.

It is to be noted that the author's definition of $Si(z)$ differs from the one usually adopted, and that his function is not an analytic function of $z=x+iy$. However, if we let $f(z)$ denote the usually accepted function $Si(z)$, with the lower limit of the integral in (2) at the origin, then

$$f(z) = Si(z) + \frac{1}{2} [\text{Ei}(y) - \text{Ei}(-y)]i.$$

Thus $f(z)$ can be obtained rather simply from $Si(z)$; the five functions are identical when z is real.

The following three tables are given:

Table I: $\text{Re}(Ci(x+iy))$, $\text{Im}Ci(x+iy)$

Table II: $\text{Re}Si(x+iy)$, $\text{Im}Si(x+iy)$

Table III: $Sia(x,y)$, $Cia(x,y)$.

All three tables are given for $x,y = 0(.1)3.1$, with the origin excluded. Entries are to 12D, but only 10D are guaranteed.

There is a useful Introduction with some of the basic properties of the functions and asymptotic expansions.

G. BLANCH

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1138[L].—T. M. CHERRY, "Tables and approximate formulae for hypergeometric functions, of high order, occurring in gas-flow theory," R. Soc. London, *Proc.*, v. 217A, 1953, p. 222-234.

If trans-sonic flow is discussed by means of the hodograph method, the functions

$$\begin{aligned}\chi_\nu(\tau) &= \tau^{1/\gamma} F(\nu - a_s, \nu - b_s; \nu + 1; \tau) \\ \psi_\nu(\tau) &= \tau^{1/\gamma} F(a_s, b_s; \nu + 1; \tau)\end{aligned}$$

are needed. Here a_s and b_s are determined by

$$a_s + b_s = \nu - \frac{1}{\gamma - 1}, \quad a_s b_s = -\frac{\nu(\nu + 1)}{2(\gamma - 1)}, \quad a_s > b_s,$$

F is Gauss' hypergeometric series, and the adiabatic exponent, γ , is taken to be 1.4 in this paper.

The point $\tau_s = (\gamma - 1)/(\gamma + 1) = 1/6$ is a transition point of the differential equations satisfied by χ_ν and ψ_ν , and in an interval including this point there is no asymptotic representation in terms of elementary functions. Another difficulty connected with these functions is caused by the fact that they are not defined when ν is a negative integer. For this reason the author introduced certain logarithmic solutions $\chi_{\nu_0}(\tau), \psi_{\nu_0}(\tau)$ which exist for positive integer ν (except ψ_0 for $\nu = 1$), and which are analogous to Bessel functions of the second kind.

The functions $\chi_\nu(\tau), \psi_\nu(\tau), \chi_{\nu_0}(\tau), \psi_{\nu_0}(\tau)$ were investigated in a previous paper by the author;¹ in that paper there are also references to the literature.

Tables 1 and 2 (p. 227-233) of the present paper give numerical values of certain slowly varying auxiliary functions sufficient to calculate $x_s(\tau)$, $\psi_s(\tau)$, $xy_s(\tau)$, $\psi_{s'}(\tau)$, $x'_s(\tau)$, $xy'_{s'}(\tau)$ for $\tau = .08 (.02).30$, $v = 10.5 (1)30.5$. The entries are given to 6S for $v \leq 20.5$, and to 4S, $v > 20.5$. These tables supplement earlier tables referred to in *MTAC*, v. 3, p. 522 (RMT 676) and v. 6, p. 30-31 (RMT 953). Table 3 (p. 233) gives 7D values of

$$\frac{\Gamma(1+a_s)\Gamma(v-b_s)2\pi\delta^{2s}}{\Gamma(1+a_s-v)\Gamma(-b_s)\Gamma(1+v)\Gamma(v)}$$

for $v = 10.5 (1)30.5$, together with a simple approximation which is good to 8D when $v \geq 20$.

If $v > 30$, the auxiliary functions of Tables 1 and 2 may be approximated by algebraic functions. The coefficients in these algebraic functions are tabulated in Tables 4 and 5, p. 233-234.

A. E.

¹ T. M. CHERRY, "Asymptotic expansions for the hypergeometric functions occurring in gas-flow theory," R. Soc. London, *Proc. v. 202A*, 1950, p. 507-522.

1139[L].—GEOFFREY KELLER & MARY FENWICK, "Tabulation of the incomplete Fermi-Dirac functions," *Astrophys. Jn.*, v. 117, 1953, p. 437-446.

The function tabulated here is

$$F(\eta, u) = \int_0^u (e^{x-\eta} + 1)^{-1} x^{1/2} dx,$$

and the authors give 3, 4, or 5 S values for $\eta = -2 (.5) 10$, $u = 0 (.2) 1 (.1) X_v$, where X_v is that value of u past which $F(\eta, u)$ remains constant (to the degree of accuracy chosen). X_v runs from 6.8 (for $\eta = -2$) to 19.5 (for $\eta = 10$). The authors believe that the maximum error in the entries is less than two units in the last place.

The authors give approximations in terms of the probability integral for $\eta < -2$, and in terms of the functions

$$G_0(v) = \int_v^\infty (e^y + 1)^{-1} dy = \log(e^{-v} + 1), G_n(v) = C_n \int_0^v (e^y + 1)^{-1} y^n dy, \\ C_1 = 1/2, C_2 = 1/8, C_3 = 1/16, \dots,$$

for $\eta > 10$. In the latter case $u \leq \eta$ and $u \geq \eta$ have to be treated separately. A 3-4D table of $G_n(v)$ for $n = 0(1)3$, $v = 0(.1)10$ is appended.

A. E.

1140[L].—H. LOTTRUP KNUDSEN, *Bidrag Til Teorien For Antennesystemer Med Hel Eller Delvis Rotations-symmetri*. I Kommission Hos Teknisk Forlag, Copenhagen, 1953, 228 p.

The author gives tables of $J_n(x) = \int_0^x J_n(t) dt$, $n = 0, 1, 2, \dots, 8$, $x = 0(.01)10$, 5D and a rough graph of $J_n(x)$, $x = 0(.5)10$. $J_n(x)$ was computed recursively using values of $J_0(x)$ (10D, $x = 0(.01)10$) tabulated by LOWAN & ABRAMOWITZ.¹ A table of $\frac{1}{2}J_0(x)$, 7D, $x = 0(.02)1$, is given by WATSON.²

In addition, there is a table of integrals of the form $\int_0^x J_n(t) \sin \alpha t dt$,
 $\alpha = t, x - t$.

DONALD RUBIN

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¹A. LOWAN & M. ABRAMOWITZ, "Table of integrals $\int_0^\infty J_0(t) dt$ and $\int_0^\infty Y_0(t) dt$," *J. Math. Phys.*, v. 22, 1943, p. 1-12.

²G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2d ed., Cambridge, 1944, p. 752.

1141[L].—E. KREYSZIG, "Der allgemeine Integralkosinus $Ci(z, \alpha)$," *Acta Math.* v. 89, 1953, p. 107-131.

The generalized sine integral was investigated in an earlier paper¹ by the same author. The present paper contains the corresponding investigation of the generalized cosine integral

$$Ci(z, \mu) = \int_0^z t^{-\mu} \cos t dt \quad \operatorname{Re} \mu < 1.$$

Table 1 (p. 120) gives 3D values of $Ci(x, \alpha)$ for $x = 0(.2)4(.5)20$, $\alpha = .25, .5, .75$.

Table 2 (p. 121-123) gives 2D or 3S values of the real and imaginary parts of $Ci(x + iy, \alpha)$ for $x = 0(1)20$, $y = 0(1)5$, $\alpha = .25, .5, .75$.

Table 3 (p. 124) gives real and imaginary parts of the first three pairs of simple zeros of $Ci(z, \alpha)$ for $\alpha = .25, .5, .75$.

Table 4 (p. 125) gives those values of $\alpha = \omega(n)$, to 4D, for which a double zero occurs at $x = \left(2n + \frac{3}{2}\right)\pi$, $n = 0(1)10$.

Table 5 (p. 125) gives 5D values of $Ci(\infty, \alpha)$ for $\alpha = .001(.001).02$ (.005).045.

This paper, as its predecessor, is accompanied by relief diagrams, altitude charts, and a bibliography. Between them, the two papers give an adequate picture of the incomplete gamma function in the complex plane, for the three selected values of α .

A. E.

¹E. KREYSZIG, "Ueber den allgemeinen Integralsinus $Si(z, \alpha)$," *Acta Math.*, v. 85, 1951, p. 117-181. *MTAC*, v. 5, p. 156.

1142[L].—MARCEL MAYOT, "Tables de fonctions intervenant dans le calcul des corrections de diffusion dans la photométrie de la lumière du ciel nocturne," *Ann. Astrophysique*, v. 15, 1952, p. 374-382.

The integrals

$$\Phi_i(\tau) = \int_0^{i\pi} (1 - \alpha^2 \sin^2 \theta)^{-\frac{1}{2}} e^{-\tau \sec \theta} \sin \theta \cos^i \theta d\theta$$

are tabulated to 4D for $i = 0, 1, 2$, $\tau = 0 (.05).7$, $\alpha = R/(R + H)$, $R = 6370$, $H = 0, 50, 100, 200, 400, 800, \infty$.

Several series expansions of these integrals are obtained, and the method of computation is described.

A. E.

1143[L].—WASAO SIBAGAKI, *Theory and Applications of the Gamma Function, with a Table of the Gamma Function for Complex Arguments Significant to the Sixth Decimal Place.* (Japanese.) 202 p., Tokyo, Iwanami Syoten, 1952. 18.5 × 25.5 cm.

The table gives 6D values of the real and imaginary parts of $\ln \Gamma(x + iy)$ for $x = -10(.2) - 6(.1)10.4$, $y = 0(.1)2(.2)10$. For a previous tabulation of $\ln \Gamma(x + iy)$ see RMT 234 (*MTAC*, v. 2, p. 19), for other tables of the gamma function in the complex domain see RMT 855 (*MTAC*, 5, p. 25–26).

A. E.

MATHEMATICAL TABLES—ERRATA

231.—(1) F. CALLET, *Tables Portatives de Logarithmes*, Paris 1795 and many later editions.

(2) F. MASERES, *Scriptores Logarithmici*. London 1796, v. 3, p. 119–123.

(3) H. M. PARKHURST, *Astronomical Tables*, New York 1868, 1889.

(4) J. PETERS & J. STEIN, *Zehnstellige Logarithmentafel*, Band 1, Berlin 1922. *Anhang*. Table 14b, p. 156–161.

The tables referred to are the 61D common logarithms of primes between 100 and 1098 originally calculated by ABRAHAM SHARP. Seven errors occurring in all four tables have been noted by UHLER¹ as the result of an extensive examination of (4). Three of these are last figure errata.

Only five of these seven errata occur in Sharp's table of 1717, his value for $\log 1097$ being correct. [*MTAC*, v. 1, p. 58, v. 7, p. 171, and R. C. ARCHIBALD, *Mathematical Table Makers*, p. 73].

¹ H. S. UHLER, "Omnibus checking of the 61-place table of denary logarithms compiled by Peters and Stein, by Callet and by Parkhurst," *National Acad. Sci. Proc.*, v. 39, 1953, p. 533–537.

232.—RUEL V. CHURCHILL, *Modern Operational Mathematics in Engineering*, 1944, p. 296, eq. 33.

For the Laplace transform pair

$$\frac{s^n}{(s^2 + a^2)^{n+1}} \subset \frac{t^n \sin at}{2^n a n!}$$

read

$$\frac{s^n}{(s^2 + a^2)^{n+1}} \subset \sum_{\mu=0}^n A_{n,\mu} t^{n-\mu} \sin \left(at + \frac{\pi \mu}{2} \right)$$

where

$$A_{n,\mu} = \sum_{\nu=0}^{\mu} \frac{(-1)^{\nu} a^{-(\mu+1)} (n+\mu-\nu)!}{2^{n+\mu-\nu} (n-\nu)! (\mu-\nu)! (n-\mu)! \nu!}$$

$$A_{n,0} = \frac{1}{2^n n! a}$$

$$A_{n,1} = \frac{-(n-1)}{2^{n+1} (n-1)! a^2}$$

$$A_{n,2} = \frac{n^2 - 5n + 2}{2^{n+2} (n-2)! 2! a^3}$$

$$A_{n,3} = \frac{-(n-3)(n^2-9n+2)}{2^{n+3}(n-3)!3!a^4}$$

$$A_{n,4} = \frac{n^4 - 22n^3 + 131n^2 - 206n + 24}{2^{n+4}(n-4)!4!a^5}$$

etc.

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Radio Corporation of America
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UNPUBLISHED MATHEMATICAL TABLES

- 171[F].—F. GRUENBERGER,
- Lists of Primes*
- . Tabulated from punched cards.
-
- Deposited in the UMT FILE.

The list of primes is from 20 000 003 to 20 040 049 and contains 2390 primes.

F. GRUENBERGER

Univ. of Wisconsin
Madison, Wis.

- 172[L].—NATIONAL PHYSICAL LABORATORY,
- Tables of Multhopp's Influence Functions*
- . 72 foolscap pages + 3 pages of description. Deposited with the ROYAL SOCIETY (no. 16).

The following integrals occur in MULTHOPP's aerodynamic theory of wing loading:

$$i(X, Y) = 1 + \frac{1}{\pi} \int_0^\pi \frac{(1 + \cos \phi)(2X - 1 + \cos \phi)}{\sqrt{(2X - 1 + \cos \phi)^2 + 4Y^2}} d\phi$$

$$j(X, Y) = \frac{4}{\pi} \int_0^\pi \frac{(2 \cos^2 \phi + \cos \phi - 1)(2X - 1 + \cos \phi)}{\sqrt{(2X - 1 + \cos \phi)^2 + 4Y^2}} d\phi$$

$$ii(X, Y) = \int_{-\infty}^X i(t, Y) dt$$

$$jj(X, Y) = \int_{-\infty}^X j(t, Y) dt.$$

Four-decimal values, within a final unit, are tabulated in the half-plane $Y \geq 0$. For convenience of arrangement and economy of space, X and Y are replaced by R and ψ , where

$$\begin{aligned} R^2 &= (2X - 1)^2 + 4Y^2 & X &= \frac{1}{2} + \frac{1}{2}R \cos \psi \\ \tan \psi &= 2Y/(2X - 1) & Y &= \frac{1}{2}R \sin \psi \end{aligned}$$

Values are given at the pivotal points

$$\begin{aligned} \psi &= 0(1^\circ)180^\circ \\ R &= .2(.05)2, 1/R = 0(.05).5. \end{aligned}$$

The function $ii(X, Y)$ which becomes infinite with R , is replaced near $1/R = 0$ by the function $ii(X, Y) - \frac{1}{2}R(1 + \cos \psi)$.

Except in certain exceptional regions near $\psi = 0$ and $\psi = 180^\circ$ where no

provision is made for interpolation, the table is interpolable using second differences. Most of these are given.

173[L].—NATIONAL PHYSICAL LABORATORY, *Tables of*

$$\frac{J_0(x)}{H_0^{(2)}(x)} + \sum_{n=1}^{\infty} 2(-1)^n \frac{J_n(x)}{H_n^{(2)}(x)} \cos n(\pi - \theta) \quad \text{and}$$

$$\frac{J_0'(x)}{H_0^{(2)'}(x)} + \sum_{n=1}^{\infty} 2(-1)^n \frac{J_n'(x)}{H_n^{(2)'}(x)} \cos n(\pi - \theta).$$

10 foolscap pages. Deposited with the ROYAL SOCIETY (no. 19).

Two-decimal values of the real and imaginary parts are given for $x = 0(.2)2(.5)5(1.0)10$; $\theta = 0(.5^\circ)5^\circ, 10^\circ(10^\circ)180^\circ$. A second table gives these functions in polar form $r e^{ia}$ (r to 2D, a to $.1^\circ$).

These functions occur in the theory of the reflexion of electromagnetic waves from infinite cylinders.

174[L].—NATIONAL PHYSICAL LABORATORY, *Table of an Integral used in Calculating Profiles of Water Waves.* 6 quarto pages. Deposited with the ROYAL SOCIETY (no. 20).

Writing $F = \int_0^{\pi/2} \sec^3 \theta \cdot e^{-\alpha^2 \sec^2 \theta} \cos(\beta \sec \theta) d\theta$, the four tables give three-decimal values of the functions indicated below. Modified second differences are given.

Table I	$2\alpha^2 F$	$\alpha = .2(0.05)1$	$\beta = 0(1)1.5$
Table II	F	$\alpha = .2(0.05)1$	$\beta = 1.5(1)3.0$
Table III	F	$\alpha = .2(0.05)1$	$\beta = 3.0(1)6.0$
Table IV	F	$\alpha = .3(1)1$	$\beta = 6.0(1)60\alpha^2$

This integral is used in calculating wave profiles and the wave resistance due to a ship's motion.

175[L].—D. H. SHINN, *Tables of Fresnel's integrals.* vii + 28 foolscap typescript and manuscript pages. Deposited with the ROYAL SOCIETY (no. 11).

Tables I and II give the functions $C(u)$ and $S(u)$, defined in UMT, 166, to 5D for $x = \frac{1}{2}\pi u^2 = 0(.01).2(.02)1(.05)20$ and for $u = \sqrt{2x/\pi} = 0(.01).4$.

Table III gives 5D values of $R(u)$ and $\frac{1}{2}\pi u^2 = \theta(u)$ where

$$R(u) = [\{C(u) - \frac{1}{2}\}^2 + \{S(u) - \frac{1}{2}\}^2]$$

and

$$\theta(u) = \tan^{-1} \frac{C(u) - \frac{1}{2}}{-S(u) + \frac{1}{2}}$$

for $t = 1/u = 0(.01).3$.

176[L].—J. K. SKWIRZYNSKI, *Tables of the error integral of a complex variable.* 5 typescript foolscap pages + 2 pages of diagrams. Deposited with the ROYAL SOCIETY (no. 12).

These tables give real and imaginary parts of the function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt = \frac{2i}{\sqrt{\pi}} \int_{iz}^{i\infty} e^{t^2} dt$$

to 4D, where $z = ae^{ib}$ for $a \doteq 0(0.05)1.3(.1)1.5$ and $b = 0(5^\circ)45^\circ$. There is also a brief introduction indicating the formulae used for the construction of the tables, and two pages of diagrams exhibiting the results.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 415 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

THE CIRCLE COMPUTER

The Circle Computer is a fully automatic electronic digital computer. It has a memory of 1024 words, each consisting of forty binary digits plus two binary digits for sign. Single address instructions are used; instructions are stored in the same memory as numbers, with two instructions in each word. The memory and the operating registers appear on a magnetic drum rotating at 3540 rpm. Input and output to the computer is obtained by parallel operation of electric typewriters and punched paper tape.

In actual computation, the Circle Computer will be several hundred times as fast as a human computer using a desk calculating machine. Eight digit decimal numbers with sign can be read into or out of the Circle Computer, with the necessary conversion from or to binary, at a rate of one a second. This speed is primarily determined by the typewriter or tape handling devices. The conversion from decimal to binary or the reverse takes negligible time.

Historical. The Circle Computer was originally conceived and designed by people whose primary interest was in having such a machine for their use. Thus, the major emphasis in the design was to obtain a machine suitable for a medium size highly technical laboratory. Further, this machine had to be operable by physicists or mathematicians or engineers without an elaborate staff or specially trained computer operators. As time went on and it seemed that this goal might be obtained, it also became clear that such a machine would be of interest to many people. In order to further this aim, it was decided that it would be desirable to find an electronic equipment manufacturer to whom the construction of electronic equipment with a few thousand vacuum tubes was an old story holding few terrors. (The Circle Computer has several hundred vacuum tubes.) Such a manufacturer was found in Hogan Laboratories, Inc. Arrangements were made for Hogan Laboratories to offer Circle Computers for sale. Three such machines are now in process of manufacture.

It may be worthwhile to state again that the Circle Computer was designed from a functional standpoint by people whose main interest is in using it, whereas it is being designed from a circuit standpoint and built by people whose business it is to build electronic equipment of the same or greater complexity.

Purpose. In recent times, most calculation has been carried out on desk calculating machines, problems ranging in size from a few man minutes to several man months. The Circle Computer is intended to be used for a

similar range of problems, extending perhaps to many man years on the large end. An attempt has been made to design the computer so that it may also be conveniently used for relatively short problems—in the extreme case, it is possible to code the computer to operate similarly to a desk calculator, but with almost any sort of special characteristics desired. In short, the Circle Computer is a general purpose computing machine, with no a priori limitations on the problems for which it is suitable.

Logical Design and Instruction Code. The logical design of the Circle Computer was patterned after the more successful operating large scale electronic digital computers. To indicate in a few words the type of computer being talked about we shall refer to it as an IAS (Institute for Advanced Study) type machine.

An IAS type machine may be thought of as consisting of three parts—arithmetic organ, memory, and control. The arithmetic organ is capable of the same sort of operations as an ordinary desk calculation machine. The memory is used to store both instructions and numbers. The control, in addition to including the usual arithmetic operations, must also contain such logical orders as comparison and the ability to modify its own coding. The extreme flexibility obtained by storing orders and numbers in the same memory and by being able to modify the instructions via the arithmetic organ has not been obtained in computing machines other than those following the IAS type logical design as just outlined.

Once the IAS type logical design has been selected, there remains the option of choosing single address, three address, or four address instruction code. Again our choice was for the somewhat greater flexibility inherent in single address coding, as compared with three or four address coding. This is greatly enhanced in the Circle Computer, in so far as modifications of the standard design are being considered. That is, it is easier to construct a special purpose single address order than a special purpose three address order. A few of the possible modified instructions are mentioned at the end of this paper.

Special Features. The instruction code of the Circle Computer is given in the Table accompanying this article along with average operating times. As is implied above, in the main the instructions are patterned after the IAS type machines. There are some additional instructions, however, which are desirable because of the lower speed and therefore slightly different emphasis of the Circle Computer design.

We now describe the instructions in the Table, with special emphasis on those not always present in IAS type machines. In a single address code, the instruction is stored in the memory in a position called the "floating address." The control of the machine changes the floating address by unity each time it carries out an order, so that the computer goes through its orders in sequence in the memory. This operation is modified by transfer orders, which change the floating address.

The arithmetic instructions of the Circle Computer are conventional for an IAS type machine.

The customary transfer operations, unconditional transfer and conditional transfer, are included, as is the rarer overflow transfer. Unconditional transfer means change the floating address to the address part of the in-

struction. Conditional transfer calls for a change in floating address only if the number in the accumulator (that register on the drum which contains the results of all arithmetic operations except division) is negative. Overflow transfer calls for a change in address if the magnitude of the number in the accumulator is too great to be handled in the Circle Computer—specifically, if the magnitude of the number in the accumulator is greater than or equal to unity, but less than two, the floating address is changed. The overflow transfer is useful in sensing the size of numbers in order to code, where necessary, floating binary point operation.

A further transfer order, as far as we know not available in any other computer, is the Function Table order. This order stores the floating address in the memory of the computer at the position called for as part of the instruction and then changes the floating address so that the control transfers to a position in the memory adjacent to that where the previous floating address was stored. The purpose of this order is to enable the utilization of subroutines in a problem by a single instruction in the main routine.

In order to call for a subroutine in coding, one must not only transfer the control to the position in the memory where the subroutine begins, but must also be sure that after the subroutine is finished, the control is returned to the next position in the main sequence. Both of these requirements are met with a single Function Table instruction. Moreover, since the floating address stored in the Function Table order is stored in the memory, there is no limitation on the number of Function Table orders that may be used, so that one can have subroutines of subroutines, and so forth. Clearly, with the Function Table order, the operator of a Circle Computer can write his own instruction code.

Since the Circle Computer operates in the binary number system whereas the input and output data will normally be in the decimal number system, binary-decimal and decimal-binary conversion is needed. These could, of course, be obtained as coded subroutines. However this would slow up the handling of input and output data. Therefore, special instructions are included in the Circle Computer commands to give high speed conversion. The input conversion converts eight digit decimal numbers to binary as a separate operation on the number after it has been read from the tape. The output conversion command is part of the print order and will convert the binary number in the accumulator to decimal as the number is printed. As many decimal digits as desired may be obtained. Thus, the output conversion takes no extra time; the input conversion takes 6% as much time as reading the input tape, which is negligible.

Communication. Input to the Circle Computer is by means of a six-hole punched paper tape operating in parallel with an electric typewriter. This input information is called for by the computer via the feed order. It can be accepted as numerical information in the hexadecimal system, in which case only four holes on the tape are relevant, or as alphabetical information utilizing all six holes. This information can be read into the computer at a rate of ten digits per second.

The above feed order reads information into the register of the computer, where it is available as numerical information. In addition, tape information can be used to select a floating address and transfer the control. In this man-

ner, any one of forty-four different subroutines can be called for by tape. Furthermore, whenever the machine is idling, the same set of subroutines can be called for by depressing keys on the typewriter. It is by means of this feature that the Circle Computer could be operated as a special purpose desk computer, for example.

The output from the computer again goes to a typewriter and tape in parallel. This output may be either in converted binary or directly in groups of four binary digits (hexadecimal) or in groups of six binary digits (alphabetical). The last possibility is useful in enabling the machine to operate the typewriter—line spacing, tabulation, even typing of headings, all being obtainable by proper coding of the computer.

The above discussion was concerned primarily with communication with the computer during normal operation. To some extent one may consider the communication with the computer when hunting for errors in its instruction code as being more important. At such a time, the operator would like to see what the computer does in some detail, so that he may see where things go wrong in the coding. In order to fill this need, monitor operation is provided in the Circle Computer. When the monitor operation switch is set to monitor, every order in the computer which occurs in the left half of a word with a negative sign is automatically followed by a Function Table order to the zero position in the memory. The operator may code such a subroutine starting at this position as will cause the computer to print out that information he wishes displayed at each step being monitored.

It is our belief that the ease and flexibility of operator machine communication as indicated above makes it possible to use the Circle Computer not only for long, tedious problems, but also for many of the smaller problems which arise continuously in a technical organization.

Physical Description. The computer proper occupies about the same floor area as an office desk. The control panel and input output are connected to the computer by means of flexible cables. The computer's appearance is shown in the frontispiece.

The electronic components have been designed with an eye toward reliability and easy maintenance. It seems likely that one operator will be able to code, operate and maintain the computer. The circuits are designed to use standard computer tubes under conservative operating conditions. The speed of the electronic components is low enough so that no high currents are required of any tube. Conventional, standard (even old fashioned) tried and true circuits are used throughout.

The electronic components are divided into small logical units of a few tubes each which are attached to the machine with screw connections. We feel screw connections are more reliable than plugs. With this arrangement of small units, servicing can be accomplished by having a small stock of replacement units and tubes.

Alternatives. The discussion above has been concerned with the standard model Circle Computer, as is the Table. One Computer now under construction has a 4096 word memory, rather than the 1024 word memory of the standard model Circle Computer and is capable of operating with words of half length. Thus, this computer can be considered either a 4096 word forty binary digit (plus sign) computer or a 8192 word twenty binary digit (plus sign) computer. A complete catalog of other alternatives is impossible,

but a few of the possibilities which may be of interest to some users are indicated below.

Special orders of various types can be invented almost indefinitely. A few of the more interesting of these include halving and doubling operations in which the register and accumulator are a single unit. This would be useful in rebuilding words in the machine for logical operations. Similarly the equivalent of logical multiplication is available. This order consists of replacing those digits of some word in the memory which correspond to unit digits in the register by the corresponding digits in the accumulator. Another possibility, which might be useful in conjunction with tape search, is a transfer order in which the transfer of control occurs unless the number in the accumulator is identical with that in the register.

Alternative input and output possibilities are also large in number. For example, multiple input or output tapes, with the desired one selected by coding, are immediately available. Tape searching is also a possibility. It should be pointed out that the input output speed is not computer limited but tape limited and could be upped by a factor of twenty or so without change in computer design.

Circle Computer Instructions Table

Each word is arranged as shown below:

$\underbrace{S_1 S_2}_{\text{Sign}}$	$\underbrace{A_1 A_2}_{\text{order}}$	$\underbrace{A_3 \cdots A_{12}}_{\text{Left}}$	$\underbrace{A_{13} \cdots A_{20}}_{\text{Left}}$	$\underbrace{A_{21} A_{22}}_{\text{Right}}$	$\underbrace{A_{23} \cdots A_{32}}_{\text{Right}}$	$\underbrace{A_{33} \cdots A_{40}}_{\text{address}}$
digits	address	command	order	order	order	command

$x = (x_1, x_2, \dots, x_{10})$ where these are stored either at $A_3 - A_{12}$ or $A_{21} - A_{32}$. The command is indicated by digits y_1, \dots, y_7 where these are stored either at $A_{13} - A_{19}$ or $A_{33} - A_{39}$.

L or R in LSp , RSp , etc. means $y_1 = 0$ or 1. Similarly E or O in ED , OD , etc., means $y_1 = 0$ or 1. We define

$$z = 64 - 32x_6 - 16x_7 - 8x_8 - 4x_9 - 2x_{10} - y_1$$

Symbol	Average Time (milliseconds)	Operation
$x, +$	25	Clear the accumulator and add the number at memory position x into it.
$x, -$	25	Clear the accumulator and subtract the number at memory position x from it.
$x, +M$	25	Clear the accumulator and add the absolute value of the number at memory position x into it.
$x, -M$	25	Clear the accumulator and subtract the absolute value of the number at memory position x from it.
$x, h+$	25	Add the number at memory position x to the number in the accumulator. The sum appears in the accumulator.

Symbol	Average Time (milliseconds)	Operation
$x, h -$	25	Subtract the number at memory position x from the number in the accumulator, leaving the result in the accumulator.
$x, h + M$	25	Add the absolute value of the number at memory position x to the number in the accumulator, leaving the result in the accumulator.
$x, h - M$	25	Subtract the absolute value of the number at memory position x from the number in the accumulator, leaving the result in the accumulator.
x, X	45	Multiply the number at memory position x by the number in the register. The more significant half of the product appears in the accumulator and the less significant half in the register.
x, XR	45	Same as x, X , except that the more significant half of the product is rounded off.
x, \div	45	Divide the number in the accumulator by the number at memory position x . The quotient appears in the accumulator.
x, R	25	Clear the register and put the number at memory position x into it.
OOO, DB	45	Convert the decimal number in the register into binary. The result appears in the accumulator. In this order, the address must be OOO as indicated and the number in the register must have the decimal digits in positions $A_{45}-A_{40}$ as zero.
x, Q	25	Store the number in the register at position x in the memory.
x, S	25	Store the number in the accumulator at position x in the memory.
x, LSp	25	Replace the address digits and y_1 of the left order at the memory position x by the corresponding digits in the accumulator.
x, RSp	25	Replace the address digits and y_1 of the right order at the memory position x by the corresponding digits in the accumulator.
x, EH OH	17-35	Halve the number in the accumulator z times.
x, ED OD	17-35	Double the number in the accumulator z times.
x, EP OP	100	Clear the print register, double the number in the accumulator $2z$ times and operate the output typewriter by the contents of the print register, which contains the last six binary digits to have overflowed the accumulator.
x, EF OF	100	Set up the six binary digits contained in the print register from the next line on the input paper tape. Left shift the register $2z$ times. The six digits from the print register now occupy the six positions in the register immediately to the right of the end of the number that was previously in the register.

Symbol	Average Time (milliseconds)	Operation
x, PBD	100 per digit	Convert the number in the accumulator to decimal and print the sign and the first $s/2$ decimal digits.
x, RC	17	Transfer the control to the right hand order at memory position x .
x, LC	17	Transfer the control to the left hand order at memory position x .
x, LCc	17	If the number in the accumulator is negative (if $S_1 = 1$) carry out the operation x, LC . Otherwise go to the next order in sequence.
x, RCc	17	If the number in the accumulator is negative (if $S_1 = 1$) carry out the operation x, RC . Otherwise go to the next order in sequence.
x, Lof	17	If, for the number in the accumulator, S_1 is unequal to S_3 , carry out the operation x, LC . Otherwise go to the next order in sequence.
x, Rof	17	If, for the number in the accumulator, S_1 is unequal to S_2 , carry out the operation x, RC . Otherwise go to the next order in sequence.
FC	100	Read the next line on the input tape and use it to determine $x_6 - x_{10}$ and y_1 . $x_1 = x_2 = \dots = x_5 = 0$. Carry out the order x, LC or x, RC with this address.
$STOP$		Idle the machine. The operator may push the start button, in which case the machine continues with the next order in sequence. Alternatively, the operator may push a typewriter key, in which case the machine performs the FC order, but from the information given it by the typewriter key rather than from the tape.
		The operation of the next pair of orders depends on their position $U = U_1, \dots, U_{10}, U_{11}$ in the memory.
x, LFT	25	Replace the address digits and y_1 of the left order at x by the digits indicating the order position next after U . Transfer the control to the right hand order at x .
x, RFT	25	Replace the address digits and y_1 of the right order at x by the digits indicating the order position next after U . Transfer the control to the left hand order at $x + 1$.

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BIBLIOGRAPHY OF CODING PROCEDURE

28. NAVAL PROVING GROUND, Dahlgren, Virginia, *Mark III Notes*. The report is a preliminary version of an operator's manual for the Mark III computer. The report discusses the number system, the external tape memory, the input coding machine and the output printers of the Mark III which is a magnetic drum machine. The remaining half of the report is devoted to the coding of elementary mathematical functions such as division, reciprocal square root, exp., log, cos, and \tan^{-1} .
29. U.S.N. AIR MISSILE TEST CENTER, *RAYDAC Operations* (Program Specifications). The order code for the RAYDAC is given along with a description of the operation determined by each of these orders.
30. A. L. LEINER, *System Specifications for the DYSEAC* National Bureau of Standards, September 1952. This description of the DYSEAC gives its operating specifications and compares them with those of the SEAC. The instructions which the machine will execute are discussed and the special features of the machine are noted.
31. JOSEPH B. KRUSKAL, JR. *A Programmer's Description of ABEL, a Magnetic Drum Relay Computer*. The George Washington University Logistics Research Project.
32. H. UCHIYAMADA, Digital Computer Laboratory, M.I.T. *Comprehensive service routines (Whirlwind)*. This description of how to use the Whirlwind I input conversion programs for translating programs using floating addresses, automatic selection of input-output and interpretive subroutines, and automatic cycle control into binary machine form is intended for reference only.
33. UNIVERSITY OF ILLINOIS, *ORDVAC Manual 1952*. Most of this well written manual is concerned with engineering details of the ORDVAC, however, the instruction code and an illustrative problem is given and there is an excellent chapter on test routines.
34. J. H. BROWN (Univ. of Michigan), *Function of Preliminary MIDAC Input Translation Program*. This memo describes how an input translation program for the three-address MIDAC is expected to be used. An example is given.

BIBLIOGRAPHY Z

1050. ASME, *Digital and Analog Computers and Computing Methods*, Symposium at the 18th Applied Mechanics Division Conference of the American Society of Mechanical Engineers held at the University of Minnesota, June 18-20, 1953. New York, 1953, 64 p.

This collection of six papers represents a wide variety of experience in the use of both analog and digital computers. There is considerable material for comparing the merits and drawbacks of the two types with their many varieties. A few problems that have been solved are described in some detail, and many more are listed in the several papers.

"Automatic solution of mechanical problems," by E. L. HARDER, of Westinghouse Electric Corporation, presents a detailed consideration of several types of problems and the advantages of different methods of solution. While analog computers are faster for those types of problems that they can readily handle, high powered digital machines are primarily de-

signed for large problems. Pertaining to digital machines, the author states "considerable study is being devoted to making their use economical for a large volume of smaller, varied problems." Some success in this direction is indicated in the paper by S. N. ALEXANDER, National Bureau of Standards, entitled "High speed digital computers and their application to problems of applied mechanics." The NBS machine, the SEAC, does many problems, some of them of quite short duration, by "time-sharing." The changeover time "seldom exceeds ten minutes."

"Digital computer methods for solving linear algebraic equations and finding eigenvalues and eigenvectors" by D. J. WHEELER & J. P. NASH, both of the University of Illinois, will be of interest to those interested in the specific jobs that must be done in solving problems on a digital computer and the relative expense of each. Similar figures are given by Alexander and Harder in their respective papers, although the techniques of solving the problems are not gone into in as great detail. "Analog solution of beams excited by arbitrary force" by W. T. THOMSON & T. A. ROGERS, of the University of California, does a similar task for an analog computer problem, again in considerable detail.

Also included is a short, well written paper describing the essential feature of a simple differential analyser. The paper, "A synchro operated differential analyzer," by A. NORDSIECK, University of Illinois, is more than half devoted to a description of a mechanical integrator which is capable of transmitting enough torque to eliminate the torque amplifiers previously needed by such machines. Three such machines are now in use on a wide variety of problems.

C. J. SWIFT

NBSCL

1051. DAVID R. BROWN & ERNST ALBERS-SCHOENBERG, "Ferrites speed digital computers," *Electronics*, v. 26, April 1953, p. 146.

Some details of the coincident current magnetic memory scheme are described in this paper, together with nominal specifications and testing procedures for the magnetic material. The scheme itself, based on the square hysteresis phenomenon which is found in highly oriented metal magnetic materials, and also in the ferrites, involves running two currents through a number of toroids arranged in a cross matrix. Only at the one toroid through which both currents pass is there enough power to switch the core from one state to the other.

The main problem in this type of memory still seems to be in obtaining a material which has a square enough hysteresis loop. If one uses the ferrite described in this paper (General Ceramics and Steatite Corp. MF-1118) access times in the range of one to 10 microseconds are practical, but the number of words per array is limited. The article does not state how many words are possible, but later information indicates that about 1000 words per array can be achieved using ferrites in this manner. This type of memory has many desirable features (low cost and ruggedness in particular), and further work will certainly improve its characteristics.

A. W. HOLT

NBSEC

- 1052.** R. D. ELBOURN & R. P. WITT, "Dynamic circuit techniques used in SEAC and DYSEAC," The Institute of Radio Engineers, *Transactions*, v. EC-2, 1953, p. 2-9.

It is claimed that all the high-speed arithmetic and control circuitry of an electronic digital computing machine to operate at a pulse repetition frequency of one megacycle can be built from just two types of etched-circuit plug-in packages. These packages, their circuits, components, and operating characteristics are described in this article.

J. H. WEGSTEIN

NBSCL

- 1053.** J. H. FELKER, "Arithmetic processes for digital computers," *Electronics*, v. 26, March 1953, p. 150-155.

This article explains how numbers may be represented in various notations by using a different radix or base, and furthermore why the binary number system is a natural choice for electronic digital computers. The basic elements of binary arithmetic are explained and amply illustrated, and the method of conversion between decimal and binary notation is also covered. Common methods of error detection, and the subject of self correcting codes are discussed briefly.

SIDNEY GREENWALD

NBSEC

- 1054.** JOINT AIEE—IRE COMPUTER CONFERENCE, *Review of Electronic Digital Computers*, 114 p. AIEE, S-44, February 1952, 28 X 21.4 cm. Price \$3.50.

This Computer Conference, held in Philadelphia, Pennsylvania, December 10-12, 1951, was arranged by a joint committee of the Committee on Computing Devices of the American Institute of Electrical Engineers and the Electronic Computers Committee of the Institute of Radio Engineers, under the chairmanship of J. C. MCPHERSON, of the International Business Machines Corporation. Joining in the sponsorship of the Conference was the Association for Computing Machinery.

The keynote address was delivered by W. H. MAC WILLIAMS, of the Bell Telephone Laboratories. It was pointed out that the Conference was arranged for the purpose of (1) a review of the useful results obtained from operating high-speed digital computers and (2) a comparison of the logic of such computers. It was hoped that assessment of the adequacy of design of high-speed digital computers in operation would point out the direction desirable for future development. The following papers were presented:

The UNIVAC System, J. PRESPER ECKERT, JR., JAMES R. WEINER, H. FRAJER WELSH and HERBERT T. MITCHELL, all of the Eckert-Mauchly Computer Corp. Division of the Remington Rand Corp.

Block diagrams of the UNIVAC, the Unityper and the Uniprinter were presented and explained. There was treated, in detail, the functioning of the UNIVAC to perform the basic arithmetic and control operations. Checking circuits and engineering features of the system aimed at reliability of operation were explained. Applicability of the UNIVAC System to scientific, statistical, commercial and logistical problems was discussed. The paper

ended with an evaluation of UNIVAC design with respect to reliability of operation, following comments on the performance record of the computer, both in acceptance testing and normal, productive operation.

Performance of the Census UNIVAC System, by J. L. MCPHERSON, Bureau of the Census, and S. N. ALEXANDER, the National Bureau of Standards.

The user's viewpoint on UNIVAC performance was given. The acceptance testing of the UNIVAC System (in this instance, a UNIVAC computer, four Uniservos, a Uniprinter and a Card-to-tape Convertor) was discussed in detail. Performance of the Computer system in the acceptance testing was outlined. A resume of operating experience on the Bureau of Census UNIVAC system was included, based on use in the Second Series Population Report problem consisting of four main parts: tallying, merging, determination of dispersion and summarizing. On a 7-day per week schedule, 24 hours a day, and during the periods from June 20 to 26, July 8 to August 4, and August 13 to October 28, 1951, the UNIVAC System was available for use 59 per cent of the time.

This level of performance was said to indicate that the Bureau of Census UNIVAC System would accomplish work planned for it at about one half the cost of doing it with any other tool available.

The Burroughs Laboratory Computer, by G. G. HOBERG, of the Burroughs Adding Machine Company.

This computer is a magnetic-drum, electronic machine, incorporating static magnetic registers. Designed and constructed in 9 months time, and checked out within 48 hours after the mounting and inter-connection of its units, the computer utilizes, wherever feasible, the Burroughs unit-packaged electronic pulse circuits. The computer is not a neatly-packaged commercial machine, however, but rather a laboratory device used in the research and development program of the Burroughs Adding Machine Company. It was described and evaluated by Mr. Hoberg on the basis of its performance in the latter role.

IBM Card-Programmed Calculator, by J. W. SHELDON and LISTON TATUM, IBM.

This paper was concerned primarily with the programming and over-all operation features of the IBM Card-Programmed Electronic Calculator. The built-in operations were discussed, with use of a control panel layout, a block diagram giving the interconnection of the units of the calculator and a slide of a typical instruction card. The general-purpose nature of the calculator was stressed. An impressive list of applications of the device, classified as to mathematical structure, together with an indication of the fields in which the problems originated, was given as a demonstration of the utility of the Card-Programmed Calculator as a design and research tool.

The ORDVAC, by R. E. MEAGHER & J. P. NASH, University of Illinois, with introductory remarks by H. H. GOLDSTINE, IAS.

The ORDVAC is a general-purpose computer built by the University of Illinois for the Ballistic Research Laboratories, Aberdeen, Md.

Dr. Goldstine's introduction was particularly relevant in view of the fact that the ORDVAC design, in many respects, follows closely the IAS Computer-Project design of a single-address, asynchronous machine with

direct-coupled circuits. The introduction was brief, consisting of a few words on the history of automatic computation and a brief classification of the type of machine represented by the ORDVAC according to number representation (binary), mode of operation (parallel) and kind of storage (Williams' tubes).

The ORDVAC paper, supplemented by slides of computer views and circuit elements, was a description and objective evaluation of design features of the device under the headings: Arithmetic Unit, Input-Output, Memory, Control and Power Supplies. Included also was a description of the operation of the ORDVAC on test routines; at the time of the Conference the ORDVAC had not been used on mathematical problems.

Design Features of the ERA 1101 Computer, by F. C. MULLANEY, Engineering Research Associates Division of Remington Rand, Inc.

This computer, a single-address, binary-system, parallel machine using magnetic-drum storage, was described functionally by use of a block diagram of its principal elements and view of various parts. Sections of the paper on testing and maintenance and the operational record of the computer indicate, first, points of the circuit design thought to contribute to reliability and, second, a rather phenomenal, substantiating operational history.

The Operation and Logic of the Mark III Electronic Calculator in view of Operating Experience was presented by GLEN E. PORTE, United States Naval Proving Ground, Dahlgren, Va.

Considerable difficulty has been experienced in placing this, one of the more complicated magnetic-drum computers, in operation. The arithmetic unit is electronic, serial, employing the coded decimal system. Of the 4,350 internal storage capacity, 200 general registers and 150 constant registers contribute fast storage (one machine cycle); the remaining 4000 registers provide slow storage (access time of several machine cycles). There is a sequence magnetic drum, in addition to eight number storage drums. Included in the system are, also, eight magnetic tape read-record units and five separate printer units. A useful auxiliary is the "coding unit" which records an instruction sequence on magnetic tape, with the use of suitable checking features. The sequence is transferred thence to the sequence drum before the beginning of computation.

Specific causes of Mark III malfunctioning were listed in the paper, and remedies taken, or planned, were given. The logical design was evaluated, the conclusion being that, with the exception of a few points, the computer is a well-balanced machine including several outstanding features which should be perpetuated in future machines.

The University of Manchester Computing Machine, by F. C. WILLIAMS & T. KILBURN, University of Manchester, Manchester, England.

This paper was the first of two on the computer which is the culmination of a research project initiated at the Telecommunications-Research Establishment at the end of the war, and which was actively supported by the TRE after its move to the University of Manchester in January, 1947. The design of the present machine, developed in close cooperation, in detail, with Messrs. Ferranti Limited, followed the construction of three prototype computers. The unique and interesting use of storage lines on cathode-ray tubes as control and arithmetic-unit registers is explained in the paper, and the general features of the computer are given. The cathode-ray storage

consists of 10,240 binary digits, on eight cathode-ray tubes, scanned serially. Magnetic storage of 150,000 binary digits is servo-synchronized with the master oscillator. Multiplication of two 40-digit numbers requires 2.16 milliseconds; the remaining operations are faster, requiring 1.2 ms. Division is not a built-in function. The input photoelectric tape reader reads 200 5-digit characters a second. Ten characters a second is the output speed of mechanical tape punching and/or teleprinting.

The Design, Construction, and Performance of a Large-Scale, General-Purpose Computer, by B. W. POLLARD, of Ferranti Limited.

Here were presented the engineering techniques used and performance of the computer constructed by Ferranti, Ltd., and installed at the University of Manchester. Perhaps the most interesting sections are those concerning the Williams Storage system, which uses a mixture of "defocus-focus" and the "dot-dash" systems, and the magnetic-drum storage system, in which the method of synchronism of drum with the basic clock wave form generator, to positional accuracy of 1/100 degree, is described. Typical examples of industrial computations on the computer, as well as selected standard subroutines were listed. It was mentioned that out of a total of 834 hours logged on the computer since its dedication, there was 74 hours fault time, giving approximately a 90 per cent availability.

The Whirlwind I Computer, by R. R. EVERETT, Digital Computer Laboratory, Massachusetts Institute of Technology.

Again, we have the first of two papers on a computer, the first one being a description of computer characteristics and the second one being more concerned with engineering aspects of the design.

The WWI, sponsored by MIT, ONR and the USAF, is a 16 binary-digit, electrostatic-storage, parallel computer, performing 20,000 single-address operations a second. Addition takes 3 microseconds complete with carry, and multiplication requires 16 μ s on the average, including sign determination. The extremely high speed of operation is a reflection of the intended applications in control and simulation problems on the computer design.

System layout and a functional description of arithmetic, control and storage units were given. There was included, in addition, a statement of the opinion of the WWI designers on what kind of checking features could profitably be included in computer design.

The manner of operating the computer, during the interval March 14, 1951, to November 22, 1951, was described. A partial list of actual problems carried out by the Digital Computer Laboratory was given.

Evaluation of the Engineering Aspects of Whirlwind I, by Norman H. TAYLOR, Digital Computer Laboratory, MIT.

In this paper, details of gate and trigger circuitry used in WWI were given. Also, control circuitry was described, and details of construction of the MIT electrostatic storage tube were presented. There was included an analysis of the WWI performance record, with particular attention to the causes of vacuum tube and crystal diode failures, and the utility of marginal checking in facilitating useful operation of the computer. The paper ended with a listing of good, adequate and doubtful points of the Whirlwind system.

The EDSAC Computer, by M. V. WILKES, University of Cambridge.

This paper described the EDSAC, which was developed at the University of Cambridge, and has been in operation there since the summer of 1949. The computer is a binary, serial machine with ultrasonic storage. It has a precision of 16 binary digits, storage capacity of 1024 words, and uses a single-address order code. A 5-hole teleprinter tape, read by a photoelectric tape reader, provides the input, and a similar tape, or direct printing on a teleprinter, is the form of the output. In addition to a description of the computer system the paper includes discussion of interesting engineering features and an analysis of component failures.

The National Bureau of Standards Eastern Automatic Computer, by S. N. ALEXANDER, National Bureau of Standards.

This paper, the first of two on the SEAC, is an account of the history of the development of the SEAC, constructed at NBS with the support of the USAF, including a narrative of the growth of its specifications as the work progressed and discussion of its operating record. Some prominent deviations from normal engineering techniques that contributed to the SEAC design are listed. A rather impressive list of problems that have been solved on the SEAC is included.

Engineering Experience on the SEAC, by RALPH J. SLUTZ, NBS.

The use of the SEAC in evaluating components under operating conditions and in testing out new auxiliary equipment in order to obtain design experience for further development is described in this paper. As an example of the former use, the 6AN5 tube, on which relatively little experience was available when the SEAC was designed, but which had been developed partly to meet computer requirements, was made the major vacuum tube in the computer. Moreover the tube was not derated, but under certain combinations of circumstances element dissipations could reach those specified by the manufacturer. The experience with the 6AN5's, for approximately 12,000 hours of computer operation, is carefully analyzed, the conclusion reached being that this tube is very satisfactory for computer use.

The characteristics of the germanium diodes used in the SEAC are discussed also. Test specifications for both the 6AN5 tubes and the diodes are given. Preventive maintenance for the computer is described, and component reliability is discussed.

Computing Machines in Aircraft Engineering, by CHARLES R. STRANG, Douglas Aircraft Company.

The scale of present-day aircraft engineering and manufacturing was painted vividly by Mr. Strang. To give one of his illustrations, the engineering man hours devoted to the DC-6 plane, up to the first flight, totals about 1,295,000 hours, and up to the time of the talk, totaled about 3,275,000 hours. The DC-6 was a development of the DC-4 on which 3,850,000 man hours of engineering work was required. Thus a grand total of about 7,120,000 man hours was required for the development of a specific type of airplane.

The extensive use of computing equipment by the Douglas Aircraft Company was discussed. By the end of 1952 it was expected that, among the three Douglas Company plants in the Los Angeles area, the computing equipment in use would comprise: 2 IBM Defense Calculators (now called the IBM 701 Calculator), 5 Card Programmed Calculators, 1 electrical analogue computer (Constructed by the William Miller Com-

pany), 1 REAC (Reeves Instrument Company), and numerous IBM 604 electronic calculators and associated equipment. The engineering method, as applied to aircraft design, was described and comparison was made of the utility of digital and analogue computational devices. Typical design problems were discussed in some detail. Some machine design objectives were suggested, as being desirable from the user's standpoint.

A review of the Bell Laboratories' Digital Computer Developments, by E. G. ANDREWS, BTL.

With the "complex number computer" as the pioneer and with Model VI as its latest achievement BTL relay computer development has spanned the pre-electronic large-scale digital computer era. The different models are described briefly, from the functional viewpoint. They are all electro-mechanical computers using telephone system relays and teletype transmitting and recording devices as their principal apparatus elements.

The succession of BTL developments had its origin in 1938 in the mind of GEORGE R. STIBITZ, then with the Bell Laboratories as a research mathematician, with the design of Model I, the Complex Number Computer. The engineering and manufacturing of the Model I were supervised by SAMUEL B. WILLIAMS, a telephone systems design engineer. These two men played an important role in the creation of several of the succeeding BTL relay computers.

An interesting historical account was given of the publicizing of the Complex Number Computer, which multiplied and divided complex numbers and accumulated results algebraically if desired, before the Mathematical Society at Dartmouth in the fall of 1940. The computer in New York was controlled by a keyboard in the University, at Hanover, N. H. Test problems were placed on the keyboard; the computer in New York made the computation and controlled the printing of the answer on a typewriter at Hanover. This feat of remote control operation was not duplicated until 10 years later.

Of the BTL computers Models V and VI represent a highly satisfactory state of the development of electromagnetic computers; evaluated according to the following figures of merit, proposed and discussed by the speaker: dependability, ease of maintenance, ease of operation, ease of programming and machine accuracy.

The Transistor as a Digital Computer Component, by J. H. FELKER, BTL.

The natural application of transistors is to provide the gain required for communications between crystal diode circuits performing all the logic functions of the computer. In such applications the life of the transistor would appear to be equal to or better than that of the best vacuum tubes that have been made for digital computer use.

Circuit details and characteristics of a high-speed regenerative amplifier using transistors, and appearing to be especially suitable for use in serial digital computers, were given. Uses of the amplifier were suggested and building blocks proposed for a transistor digital computer.

Digital Computers—Present and Future Trends, by JAY W. FORRESTER, Digital Computer Laboratory, MIT.

The paper summarized the present status of digital computer development, enumerated the better features of the machines described at the

Conference, and indicated trends to be expected in the future. Evaluation criteria were proposed for storage performance, design efficiency and reliability. The utility of marginal testing was stressed, in pointing out design weakness in a new machine and in assuring operating away from the failure threshold of components.

In the discussion of future trends, it was pointed out that, although the transistor looks promising, it should not be considered a panacea. In fact, the transistor is not so interesting for its small size and power consumption as for the unproven possibility that it can be made to operate in greater freedom from intermittent change in performance than the vacuum tube.

In connection with the Williams tube memory, it was stated that computer design groups had probably been overoptimistic in planning to use in parallel machines, at high access rates, the tube which was designed by F. C. WILLIAMS for use in connection with serial scanning techniques. It was suggested that the electrostatic tube, regardless of type, is but a transient on the stage, to be replaced in the next few years by developments in solid state physics. It was predicted, however, that the great steps in the near future would lie in the direction of simplification of computer circuitry without the loss of performance.

Without exception, the talks at the conference were supplemented by remarks from the floor. In many cases, the speakers were led into interesting discussion by questions from the audience. Also, during the course of the sessions there were informal meetings of representatives of some of the various computer groups at the conference to discuss problems arising in programming. The conference apparently was well timed, with an agenda stimulating the interest of all participants.

E. W. C.

1055. L. B. LUECK & W. W. WETZEL, "Performance of high output magnetic tape," *Electronics*, v. 26, March 1953, p. 131-133.

Recent advances in the formulation of magnetic materials have produced an increase of more than two to one in magnetic remanence of the oxide used for magnetic recording tape. This results in a signal output of about twice that of standard tapes. The gain is achieved with no increase in noise level, thereby giving an improvement in signal-to-noise ratio. This gain can be used to reduce necessary tape speed or decrease recording track width.

JAMES L. PIKE

NBSEC

1056. OFFICE OF NAVAL RESEARCH, *Digital Computer Newsletter*, v. 5, July 1953, 11 p.

The contents are as follows:

1. UNIVAC
2. Moore School Automatic Computer (MSAC)
3. The SWAC
4. Burroughs Laboratory Computer
5. Consolidated Electronic Digital Computer, Model 30-201

6. Monrobot Electronic Calculator
7. The Oak Ridge Automatic Computer (ORACLE)
8. The NAREC
9. The ELECOM Computers
 - ELECOM 100
 - ELECOM 120
 - ELECOM 200
10. University of Illinois Computer (ILLIAC)
11. Air Force Missile Test Center Computer (FLAC)
12. Naval Proving Ground Calculators
13. The Institute for Advanced Study Computer
14. The Logistic Computer
15. ABERDEEN PROVING GROUND COMPUTERS
 - The ORDVAC
 - The EDVAC
 - The ENIAC
16. Whirlwind I
17. ERA 1103
18. Hughes Airborne Computer
19. Elliott-N.R.D.C. Computer 401 Mk 1
20. NICHOLAS

Data Processing and Conversion Equipment

1. Solid Acoustic Delay Line Memory Unit—Model 3C1-384
2. Contact Telereader
3. Character Display Signal Generator

List of Computing Services

Computer Courses

1. Remington Rand Inc. (UNIVAC Training Courses)

Notices

1. Computer Symposium
2. Joint Computer Conference
3. DCN News Item

1057. WILLIAM ORCHARD-HAYS, *The Duplex System for IBM's Model II CPC, A Fast Four Address, Double Operation, Floating-Decimal Set-Up*. Project RAND Research Memorandum 1044, The RAND Corporation, February 23, 1953, 42 p., 21.65 × 28 cm. Available without charge to libraries and research institutions.

The author described a floating decimal computing system for the Model II CPC which incorporates many desirable features. It is a four-address system with three inputs, two operations and one result per card—most pairs of operations being done in one machine cycle, i.e., while maintaining the maximum speed of 150 cards per minute. Numbers are represented by a combination of eight decimal digits and sign plus two digits for the exponent representing the scale factor.

In addition to the basic arithmetic operations, two (sometimes three)

of which can be done in one card cycle, one can compute square root, log (base 10), antilog (base 10) and cosine (radians). Log and antilog are each two card operations. The negative balance test is used as a master control to switch to an alternate control field. Ten conditions such as attempting to take the square root or log of a negative number or committing certain errors in coding will cause the machine to stop.

Complete wiring diagrams of 418 and 605 boards together with detailed explanations of both are included.

R. K. ANDERSON

NBSCL

- 1058.** N. ROCHESTER, "Symbolic programming," The Institute of Radio Engineers, *Trans.*, v. EC-2, 1953, p. 10-15.

Symbolic programming is described as a process in which instead of writing a true computer code the programmer replaces true addresses by symbolic addresses such as 1.1, 1.2, 1.10, 1.10.3 and writes out the operation in lieu of the true characters which the computer uses. With this method of programming the IBM 701 system, one instruction is punched on each card and the computer calls in the symbolic code, processes it, and punches out a true machine code ready for use. Several advantages are claimed over conventional programming. For example, instructions can be added to a program by merely inserting cards, and subroutines or library programs can be included in a program by just placing their cards with the input deck.

J. H. WEGSTEIN

NBSCL

- 1059.** C. R. STRANG, "Computing machines in aircraft engineering," *Electrical Engineering*, v. 73, Jan. 1953, p. 43-48.

This informative article points out the many and complicated problems which a modern aircraft laboratory must solve before undertaking to build an acceptable airplane model. A number of such problems are presented accompanied by their mathematical formulas or schematic diagrams. Even a cursory glance at this formidable array makes the reader appreciate the necessity for the elaborate computing equipment, listed by the author, at present in use by the Douglas Aircraft Company. This includes electronic computers, both analogue and digital.

The author concludes the article by pointing out some features which he would like computers to possess. It is not clear to the reviewer in what way the UNIVAC, or the SEAC, or the RAYDAC—to mention only three of the existing high-speed computers—fail to measure up to his specifications.

IDA RHODES

NBSCL

NEWS

The American Society of Mechanical Engineers.—The eighteenth National Applied Mechanics Division Conference of the Society was held on June 18-20, 1953, at The University of Minnesota, Minneapolis, Minnesota. Included in the Conference was a Symposium on Digital and Analog Computers and Computing Methods. The program for the Symposium was as follows:

Friday, June 19, 1953, 9:30 a.m.

Symposium I

J. ORMONDROYD, Univ. of Michigan, *Chairman*

S. LEVY, NBS, Washington, D. C., *Vice Chairman*

W. T. THOMSON & T. A. ROGERS, Univ. of California, Los Angeles

A. NORDSTECK, Control Systems Laboratory, Univ. of Illinois

G. D. McCANN & C. H. WILTS, California Institute of Technology, Pasadena

Analog solution of beams excited by arbitrary force

A synchro-operated differential analyzer

New analog computers and their application to aircraft design problems

Panel discussion on applications of analog computing equipment.

Friday, June 19, 1953, 1:30 p.m.

Symposium II

L. E. GOODMAN, Univ. of Illinois, *Chairman*
J. L. BOGDANOFF, Purdue Univ., *Vice Chairman*

D. J. WHEELER & J. P. NASH, Univ. of Illinois

S. N. ALEXANDER, NBS, Washington, D. C.

E. L. HARDER, Westinghouse Electric Corp.

Panel discussion on applications of digital computers.

The banquet at 7:00 p.m. on Friday featured as speaker Dr. MINA REES, Office of Naval Research, Washington, D. C.; the subject of Dr. Rees' talk was "Future field of application of high-speed computers."

Massachusetts Institute of Technology.—A Special Summer Program on "Digital Computers and Their Applications" was offered at M.I.T. during the Summer Session of 1953, from August 24 through September 4. The course stressed general precepts for coding of problems and attempted to reveal the advantages, difficulties, potentialities, and limitations of current electronic digital computers. In this connection, students were allowed to make use of the M.I.T. Whirlwind computer, in particular, and their experience gained on this computer was directed to all high-speed computers in general. The instruction was slanted to the selection of suitable computation methods and to ways of reducing programming time, computing time, and mistake-location time associated with coding. The program was under the direction of CHARLES W. ADAMS. The instructors were composed of staff members of the Digital Computation Laboratory, and the lecturers included M. V. WILKES, Director, University Mathematical Laboratory, Cambridge, England, and JAY W. FORRESTER, Director, M.I.T., Digital Computation Laboratory.

The University of Michigan.—From Monday, August 10, through Friday, August 21, 1953, the Willow Run Research Center of the Engineering Research Institute of the University presented a special two-week program on Digital Computers. The course was designed to emphasize the present and future applications of machines now in operation with special attention to business and industrial applications, scientific computations, digital simulation, and process control. The MIDAC (Michigan Digital Automatic Computer) was made available to the enrollees for training purposes, and the techniques and methods used on this machine were made applicable to a much wider range of computers. Special attention was given to the methods of programming of problems for machine solution originating at the various digital computer establishments throughout the world.

The instruction included discussions of number systems, machine instruction methods, computational modes, subroutines and the subroutine concept, translation processes, general aids to programming, storage devices, input-output equipment, computer operation procedures, and computer reliability. The special program was under the direction of JOHN W. CARR. The instructional staff was composed of members of the Digital Computation Group. Special lecturers were N. R. SCOTT, builder and designer of the University of Michigan magnetic drum computer, and W. F. BAUER, Willow Run Research Center, mathematician and analyst, and specialist in digital simulation.

Wayne University Computation Laboratory.—A special summer course in computer applications and components was offered by Wayne University from August 10 to August 21, 1953, in an endeavor to meet partially the immediate need for trained personnel in the field of automatic computing machinery. The course was held to train people to prepare and program problems for automatic computers, to adapt electronic techniques to the problems in business and industry, and to apply new design ideas for more effective equipment. This course is a part of a comprehensive educational program in this field instituted by the University in cooperation with local industry.

Applicants could register for any one of three different groups, namely; business applications, engineering applications, and computer components. In the first group, three daily lectures were given on programming, selected accounting applications, and production scheduling. Actual sample payroll or inventory problems were coded and run on the computers. The members of the second group joined the lecturers on coding and programming in the first group in addition to hearing separate lectures on engineering applications. Members of the group on computer components attended lectures on magnetic drum and other memory systems, ferro-electric and ferro-magnetic materials, and transistors. This special course offered as lecturers and discussion leaders many prominent leaders in the field of computers and their applications. Two new computers were available to students in order that they might code problems in their own field of interest, and with the assistance of the laboratory staff that they might run them on the machines. One was a large scale digital computer with a 5000-word magnetic drum memory, built by the Burroughs Corporation, and the other was a digital differential analyzer type of machine furnished by the Bendix Aviation Corporation.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z

- 1060.** P. B. AITKEN, "Ruler for drawing radio activity decay curves," *Nucleonics*, v. 10, no. 6, 1952, p. 64.

The curvature of the edge of this ruler is varied by driving a wedge into a slot.

- 1061.** ANON., "A high speed crystal clutch," Franklin Inst., *Jn.*, v. 252, 1951, p. 427-428.

A note on certain results of the National Bureau of Standards program for the development of fast acting clutches suitable for use in high speed computers.

- 1062.** VALENTINE APPEL, "Companion nomographs for testing the significance of the difference between uncorrelated percentages," *Psychometrika*, v. 17, 1953, p. 325-330.

1063. J. D. AYERS & J. P. STANLEY, "The rolling totals method of computing sums, sums of squares, and sums of cross-products," *Psychometrika*, v. 17, 1952, p. 305-310.

1064. E. BATSCHELET & H. R. STRIEBEL, "Nomogramm zur Bestimmung der reellen und komplexen Wurzeln einer Gleichung vierten Grades," *Zeit. für angew. Math. und Physik*, v. 3, 1953, p. 156-159.

1065. B. B. BAUER, "Transformer analogs of diaphragms," *Acoustical Soc., Jn.*, v. 23, 1951, p. 680-683.

The author discusses by means of a number of examples the use of transformer analogs for diaphragms in obtaining the equivalent electrical circuits for an electromechanical system involving transducers. He gives reasons for preferring such an analog to the "conventional" one which uses a mechanical impedance.

1066. G. A. BENNETT, "Nomogram for calculating shielding for Co⁶⁰," *Nucleonics*, v. 8, no. 4, 1951, p. 55-58.

1067. F. W. BILLMEYER, "Nomographs for converting between Hunter color difference meter readings and I. C. I. color coordinates," *Optical Soc. Amer., Jn.*, v. 41, 1951, p. 860-861.

1068. P. J. BURKE, "IBM computation of sum of products for positive and negative numbers," *Psychometrika*, v. 17, 1952, p. 231-233.

1069. J. S. CAMPBELL & D. F. WELCH, "Graphical analysis of cloud chamber photographs," *Nucleonics*, v. 10, no. 12, 1952, p. 62-64.

The graphical transformation of orthographic drawings based on stereoscopic photographs is discussed.

1070. J. M. CISAR, "Nomograph for materials irradiation," *Nucleonics*, v. 6, no. 1, 1950, p. 63-66.

1071. W. C. DAVIDON, "Nomogram for computing register losses," *Nucleonics*, v. 10, no. 12, 1952, p. 76-77.

1072. P. A. EINSTEIN, "Factors limiting the accuracy of electrolytic plotting tanks," *British Jn. Appl. Physics*, v. 2, 1951, p. 49-55.

An experimental investigation of tank errors due to polarization, mechanical misalignment and surface tension was made using special tanks. At low frequencies the impedance between two electrodes is non-linear but at higher frequencies, it becomes linear but not purely resistive, having a capacitative component. The reactive element is due to the surface impedance of the electrode, which may be minimized by a proper coating. When care is taken to eliminate known errors an accuracy of .2% is obtainable.

F. J. M.

- 1073.** H. W. GOHEEN & M. D. DAVIDOFF, "A graphical method for the rapid calculation of biserial and point biserial correlation in test research," *Psychometrika*, v. 16, 1951, p. 239-242.

A large chart is available for the stated purpose.

- 1074.** H. GULLIKSEN & L. R. TUCKER, "A mechanical model illustrating the scatter diagram with oblique test vectors," *Psychometrika*, v. 16, 1951, p. 233-236.

Model illustrates changes to oblique coordinate systems.

- 1075.** F. I. HAVLÍČEK, "Nomogram for estimating the optimum aperture of optical systems," *Optical Soc. Amer., Jn.*, v. 41, 1951, p. 483-484.

- 1076.** V. M. HICKSON, "Photo elastic determination of free boundary stress on 'frozen stress' models by an oblique incidence method," *British Jn. Appl. Physics*, v. 2, 1951, p. 261-269.

"Frozen stress" models are obtained by taking a plastic model of a part, and heating and cooling it while subject to an analogous load. The permanent changes in the optical properties of the model which occur can be utilized to determine the principal stresses. This article reviews the literature and the basic theory is given in order to indicate the effect of experimental errors on the answers obtained. The actual experimental procedure is described in detail with a discussion of methods of minimizing errors. Accuracies of 4 to 6 per cent are obtainable for the maximum stresses present.

F. J. M.

- 1077.** E. S. HODGE, "A gamma nomograph," *Optical Soc. Amer., Jn.*, v. 41, 1951, p. 731-732.

The nomograph is for calibrating spectrometers.

- 1078.** H. T. JESSOP, "The scattered light method of exploration of stresses in two and three dimensional models," *British Jn. Appl. Physics*, v. 2, 1951, p. 249-260.

This is a general article describing the process of determining the stresses in a transparent model using the interference effects associated with the scattering of a very narrow beam of plane polarized light. An observer viewing the length of such a beam would observe interference fringes whose width indicates the stress present in the path of the beam. This situation is precisely analyzed and four types of problems involving two and three dimensions are given to which the method is applicable. A discussion of various materials for the model and a description of the optical system appear. Four examples are discussed. The maximum accuracy mentioned in these is three percent.

F. J. M.

- 1079.** W. B. MILLER, JR., "Nomogram for estimating decay of I^{131} and P^{32} ," *Nucleonics*, v. 9, no. 4, 1951, p. 58-59.

1080. P. MOON & D. E. SPENCER, "Slide rule for lighting calculations," Optical Soc. Amer., *Jn.*, v. 41, 1951, p. 98-103.

Two special slide rules and directions for their use are described.

1081. P. MOON & D. E. SPENCER, "Simplified interreflection calculations," Franklin Inst., *Jn.*, v. 251, 1951, p. 215-230.

Certain problems in illumination which previously had been handled by tables are solved approximately by means of nomograms and graphs.

1082. S. C. REDSHAW, "A three dimensional electrical potential analyzer," *British Jn. Applied Physics*, v. 2, 1951, p. 291-295.

This analyzer consists of a resistance network. The network was woven of high resistance wire and nine tiers of 25 by 25 points were provided. The cubical lattice form is appropriate for flows perpendicular to a plane surface.

F. J. M.

1083. A. L. SCHOEN & R. H. DAVIS, "An alignment chart for computing the thicknesses of evaporated films," Optical Soc. Amer., *Jn.*, v. 41, 1951, p. 362-363.

1084. L. SIEGEL & E. E. CURETON, "Note on the computation of biserial correlations in item analysis," *Psychometrika*, v. 17, 1952, p. 41-43.

A method is described for the computation of biserial correlation with a large number of items using punched card equipment.

1085. A. P. SPEISER, "Rechengeräte mit linearen Potentiometern," *Zeit. für angew. Math. und Physik*, v. 3, 1952, p. 449-460.

The use of loaded potentiometers to represent functions of one or two variables is discussed.

1086. L. G. STANG & P. D. HANCE, "Nomogram for calculating fission product activities," *Nucleonics*, v. 10, no. 1, 1952, p. 48-49.

1087. RAJKO TOMOVICH, "A universal unit for the electrical differential analyzer," Franklin Inst., *Jn.*, v. 254, 1952, p. 143-151.

The unit mentioned in the title is a complicated combination of rotating switches which can successively switch fifty preset potentiometers into a circuit. This permits the representation of a function of one variable having fifty values. The device is driven by either a constant speed motor or by a servo mechanism if a function of a generated variable is desired.

F. J. M.

1088. W. WALCHER, "Graphische Methode zur näherungsweisen Bestimmung von Trägerbahnen in elektrostatischen Linsen unter Berücksichtigung des Raumladungseinflusses," *Zeit. für angew. Physik*, v. 3, 1951, p. 189-190.

- 1089.** LOUIS G. WALTERS, "Hidden regenerative loops in electronic analog computers," I. R. E. (Professional Group on Electronic Computers), *Trans.*, v. EC-2, no. 2, 1953, p. 1-4.

The author considers the linear differential equations with constant coefficients that describe a simple electrical network. The characteristic polynomial of the system is of third degree. One formal method of derivation leads to a set of 2 second order equations whose leading matrix is singular. In coding this set for electronic analog computation, it is necessary that the gain of a loop consisting of 2 amplifiers be precisely 1. The fourth order system that results if the loop gain is $1 + \epsilon$ has an extraneous mode which diverges rapidly for positive ϵ and decays rapidly for negative ϵ . The author suggests recasting the equations to avoid the singular matrix or introducing a small negative value of ϵ in the loop.

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- 1090.** BRUCE B. YOUNG, "Advanced time scale analog computer," *Franklin Inst., Jn.*, v. 253, 1952, p. 169-171.

A repeating type of differential analyzer for systems with constant coefficients. Results are displayed on an oscilloscope. Four non-linear elements are available. This device was mentioned also in an anonymous note in the same journal, v. 251, 1951, p. 488.

NOTES

- 154.—ON A COMPUTATION OF THE CAPACITY OF A CUBE.** The electrostatic capacity (transfinite diameter) of a two or three dimensional region is a domain functional to which considerable attention has been paid in the last generation. Although a number of independent approaches are known, the actual computation of the capacity for specific regions is attended by considerable numerical difficulty. The present note reports the result of a computation of the capacity of the unit cube which was recently carried out on SEAC and which employed the purely geometric definitions of FEKETE¹ and PÓLYA & SZEGÖ.²

Let M designate a three dimensional region and C its capacity. Then the following two formulas are due to Pólya & Szegö:²

$$(1) \quad C = \lim_{n \rightarrow \infty} \max_{P \in M} \left(\frac{n}{2} \right) / \left[\sum_{i < j \leq n} \frac{1}{d(P_i, P_j)} \right],$$

$$(2) \quad C = \lim_{n \rightarrow \infty} \min_{(P_k) \in M} \max_{P \in M} \left\{ n / \sum_{k=1}^n \frac{1}{d(P, P_k)} \right\},$$

where $d(P, Q)$ indicates the distance between P and Q .

Formula (2) was employed in the SEAC computation, and the maximizations and minimizations were accomplished by selecting P and P_k from a quasi-random sequence of points lying in the unit cube and monitoring the extreme values. A value $n = 8$ was used. Corresponding to a fixed selection

of points (P_1, \dots, P_s) , a total of 50 points P were probed and the maximum value of the bracketed expression in (2) selected. This maximum was then printed out and the maximizing coordinates recorded. A second selection of points (P_1, \dots, P_s) was then tried. The computation was run on SEAC for approximately 3 hours during which time 250 selections of (P_1, \dots, P_s) were obtained. This represents a total of 10^6 distances $d(P, P_k)$. The minimax obtained in this way was

$$(3) \quad C = .6835.$$

No *a priori* investigation of the significance of this result has been made, and the computation made no special use of the symmetries of M .

The best known theoretical value for the capacity C of the unit cube has been given recently by W. Gross.⁴ His value is

$$(4) \quad C = .6464 + \epsilon, |\epsilon| \leq .032.$$

The agreement between (3) and (4) is surprisingly good, but it is felt, somewhat fortuitous. This method is easily adapted to regions of irregular shape. Formula (1) avoids a minimization at the cost of computing more distances.

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¹ M. FEKETE, "Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten," *Math. Zeit.*, v. 17, 1923, p. 228-249.

² G. PÓLYA & G. SZEGÖ, "Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen," *Jn. für die reine und angew. Math.*, v. 165, 1931, p. 4-49.

³ G. PÓLYA & G. SZEGÖ, "Isoperimetric inequalities in mathematical physics," *Annals of Math. Studies*, no. 27, Princeton, 1951.

⁴ W. GROSS, "Sul calcolo della capacità elettrostatica di un conduttore," *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.*, s. 8, v. 12, 1952, p. 496-506.

155.—A METHOD OF RADIX CONVERSION. The method of radix conversion presented here is useful mainly for hand computation or for converting from octal to decimal in a decimal computer. The method depends upon the identity:

$$\begin{aligned} \left(1 + \frac{b-a}{a}\right) \cdots \left(\left(1 + \frac{b-a}{a}\right)\left(\left(1 + \frac{b-a}{a}\right)\left(\left(1 + \frac{b-a}{a}\right) u_n a^n \right.\right.\right. \\ \left.\left.\left. + u_{n-1} a^{n-1}\right) + u_{n-2} a^{n-2}\right) + \cdots + u_1 a\right) + u_0 \\ \equiv u_n b^n + u_{n-1} b^{n-1} + u_{n-2} b^{n-2} + \cdots + u_1 b + u_0, \end{aligned}$$

which can be easily verified. For $n = 2$, for example, the identity is

$$\left(1 + \frac{b-a}{a}\right)\left(\left(1 + \frac{b-a}{a}\right) u_2 a^2 + u_1 a\right) + u_0 = u_2 b^2 + u_1 b + u_0.$$

Let us consider u_i to be the i -th digit of a number of radix b . We now have the right hand member of the identity to evaluate, using the radix a . This is a simple operation since $b - a$ is an integer and a in the denominator indicates a shift. An illustrative example follows:

Let it be desired to convert the octal number 17324172 to decimal

$$\begin{array}{r}
 a = 10 \\
 b = 8 \\
 \hline
 \frac{b-a}{a} = - .2
 \end{array}
 \quad
 \begin{array}{r}
 1\ 7\ 3\ 2\ 4\ 1\ 7\ 2 \\
 -\ 2 \\
 \hline
 1\ 5\ 3 \\
 -\ 3\ 0 \\
 \hline
 1\ 2\ 3\ 2 \\
 -\ 2\ 4\ 6 \\
 \hline
 9\ 8\ 6\ 4 \\
 -\ 1\ 9\ 7\ 2 \\
 \hline
 7\ 8\ 9\ 2\ 1 \\
 -\ 1\ 5\ 7\ 8\ 4 \\
 \hline
 6\ 3\ 1\ 3\ 7\ 7 \\
 -\ 1\ 2\ 6\ 2\ 7\ 4 \\
 \hline
 5\ 0\ 5\ 1\ 0\ 3\ 2 \\
 -\ 1\ 0\ 1\ 0\ 2\ 0\ 6 \\
 \hline
 4\ 0\ 4\ 0\ 8\ 2\ 6
 \end{array}$$

Thus 4040826 is the decimal equivalent of the octal integer 17324172.

By this method $n - 1$ extractions and $2(n - 1)$ additions are required to convert the number of n octal digits to its decimal equivalent by the above method. By the other two methods in common use at least $n - 1$ extractions and $5(n - 1)$ additions are required, where 5 is taken as the average value of each digit.

The method is used to good advantage in wiring type 604 IBM computers for octal to decimal conversions. For converting an eight digit octal number it requires only 21 program steps using 21 electronic cycles. The other methods in common use require at least 24 program steps and 141 electronic cycles.

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156.—FACTORS OF FERMAT NUMBERS. On August 14 and 15, the SWAC discovered that $2^{2^9} + 1$ is divisible by $825753601 = 1575 \cdot 2^{19} + 1$ and that $2^{2^{10}} + 1$ is divisible by $45592577 = 11131 \cdot 2^{12} + 1$. These are the first factors found of the 16th and 10th Fermat numbers $F_n = 2^{2^n} + 1$. The result concerning F_{16} is of some interest, since it proves that not all numbers of the form

$$2 + 1, 2^2 + 1, 2^{2^2} + 1, 2^{2^3} + 1, \dots,$$

are primes.

The composite nature of F_{10} had been revealed on February 4, 1952, when the SWAC, using R. M. ROBINSON's routine, showed that F_{10} did not divide

$3^{1000} + 1$. The residue found by the SWAC has been checked using the modulus $11131 \cdot 2^{12} + 1$ and found to agree.

The writer's SWAC routine has tested all numbers of the form $D = (2k+1)2^r + 1$ with $D < 2^{36}$ and $k < 2^{16}$ which are possible divisors of Fermat numbers. This took $3\frac{1}{2}$ hours running time.

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CORRIGENDA

V. 7, p. 114, 1. - 1, add footnote, A. W. BURKS, H. H. GOLDSMITH, and J. VON NEUMANN, *Preliminary Discussion of the Logical Design of an Electronic Computing Instrument*, Institute for Advanced Study, June 1946.

V. 7, p. 118, 1. - 7, for W. S. MACWILLIAMS read W. H. MAGWILLIAMS.

V. 7, p. 168, l. - 8, - 9, for 5 read .5.

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- C. Logarithms
- D. Circular Functions
- E. Hyperbolic and Exponential Functions
- F. Theory of Numbers
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